IEJME OPEN ACCESS

INTERNATIONAL ELECTRONIC JOURNAL OF MATHEMATICS EDUCATION
e-ISSN: 1306-3030. 2020, Vol. 15, No. 3, em0587
https://doi.org/10.29333/iejme/7865

The 8th Grade Students' Competencies in Alternating Different Symbolic Representations of Rational Numbers

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ABSTRACT

One of the attributes of rational numbers that make them different from integers are the different symbolic modes (fraction, decimal and percentage) to which an identical number can be attributed (e.g. $\frac{1}{4}$, 0.25 and 25%). Some research has identified students' difficulty in mental calculations with rational numbers as has also the switching to different symbolic representations between fractions and decimals. However, pupils' performance, and repertoire of strategies have not been systematically studied in mental calculations with rational numbers expressed in different symbolic representations. The principal question of this research: how is the ability of students to perform mental calculations with rational numbers affected when the same number changes in fraction, decimal and percentage? For the purpose of the study 62 8th grade students were interviewed to examine how this symbolic shift in the number of operations affects the success and type of strategies they use, and the ability to alternate the rotation of these symbolisms. The results of the research show that the symbolic change of the rational numbers affects the success and the type of strategies that students use in mental calculations. Another result of the study demonstrated that students are not flexible when switching between the different symbolic representations of rational numbers as benchmark while performing mental calculations.

Keywords: mental calculation, symbolic representation, rational numbers

INTRODUCTION

Mental calculation is an essential component of number sense and leads to a better understanding of rational numbers (Lemonidis, 2015). The acquisition of number sense has been recognized as a fundamental component of learning mathematics. Important factors that determine the quality of mental calculations and number sense are flexibility and the variety of strategies that can be used. Much research (McIntosh, Reys, Reys, Bana, & Farrell, 1997; Reys, 1984; Sowder, 1990, 1992; Trafton, 1992) has examined and correlated student ability in number sense with flexibility in mental calculations. The results of these studies indicate that number sense is a fundamental condition for the development of students' flexibility in mental calculations.

Kalchman, Moss, and Case (2001) report that the characteristics of good number sense include the ability to estimate and judgments of the size, the ability to recognize unreasonable results, the flexibility in mental calculations, and the ability to represent the same number in multiple ways and use the most appropriate representation to perform a calculation.

Article History: Received 9 February 2020 ◆ Revised 12 March 2020 ◆ Accepted 12 March 2020

In terms of decimals and fractions, understanding and correct use (Sowder & Schappelle, 1989), the switch and use of different representations of the same number (Markovits & Sowder, 1994; R. Reys & Yang, 1998) are components of number sense.

As we will see below, though some research has investigated the flexibility to move between fractions and decimals in mental calculations (Reys, Reys, Nohda, & Emori, 1995; Pagni, 2004; Sweeney & Quinn, 2000), the ability in switching between the three different symbolic representations - fraction, decimal, percentage - has not been systematically investigated for the same rational number.

We chose to investigate this ability to use and interchange the three different symbolic representations of one rational number in the four operations to secondary school students who have completed their teaching of rational numbers. In addition to this research, the rational numbers we chose to examine students are rational numbers as benchmark. We chose these rational numbers because they are considered basic and are used in calculations with other rational numbers. More details about these numbers are presented below.

In the following sections, the term 'rational numbers as benchmark' is clarified. Studies on determination of conceptual and procedural strategies as well as on performance and the kind of strategies of mental calculation with rational numbers and research into the flexibility to move between fractions and decimals in mental calculations are presented; the research method is then explained. The main results are thereafter presented and extensively discussed in the conclusion.

Rational Numbers as Benchmark

Benchmarks are defined as, "a compass provides a valuable tool for navigation, numerical benchmarks provides essential mental referents for thinking about numbers", (Mclntosh, Reys, & Reys, 1992, p. 6). McIntosh, Reys, and Reys (1992) state that "benchmarks are often used to judge the size of an answer or to round a number so that it is easier to mentally process" (p. 6).

We can say that, compared to other numbers, benchmark numbers have some peculiarities such as: students calculate with them initially and more easily than the other numbers, and they are used as intermediary steps to calculate with the other numbers (e.g. 13% of 45 is 10% and 3% of 45). Lembke and Reys (1994), for example, report that in solving percentage problems in an interview, the repeated halving strategy that uses benchmark percentages such as 50%, was the most frequently employed strategy. Caney & Watson (2003, p. 11) state that 'A related issue is students' use and understanding of "benchmarks," those numbers that students appear to encounter and develop facility in using first, for example, ½, 0.5, and 50%, or 0.25, 25%, and ¼.'

The rational numbers as benchmark used in this study are: $\frac{1}{4}$, 0.25, 25%, $\frac{1}{10}$, 0.1, 10%, $\frac{1}{2}$, 0.5, 50% and $\frac{3}{4}$, 0.75, 75%.

Conceptual and Procedural Strategies

Several studies are cited with regard to strategies used by students in the mental calculation of operations with rational numbers (e.g. Callingham & Watson, 2008; Caney & Watson, 2003; Clarke & Roche, 2009; Lemonidis & Kaiafa, 2014; Lemonidis, Tsakiridou, & Meliopoulou, 2018; Post, Cramer, Behr, Lesh, & Harel, 1993; Yang, Reys, & Reys, 2009).

Skemp (1976) distinguished and contrasted two kinds of understanding, the *relational* and the *instrumental*. According to Skemp (1976, p. 20), instrumental understanding is 'rules without reasons' when the student applies an algorithmic process mechanically. Relational understanding is based on the understanding of concepts and their interconnection, 'knowing both what to do and why' (Skemp, 1976, p. 20), so that the student will know what he is doing and why he is doing it without relying simply on the application of rules.

McIntosh, De Nardi, and Swan (1994) believe that strategies can be separated into *instrumental* and *conceptual*. This division by McIntosh et al. is based on Skemp's (1976) terms of instrumental and relational understanding; McIntosh et al. replace the term *relational* by the term *conceptual*.

Caney and Watson (2003) distinguish two large categories of *instrumental* or *procedural* and *conceptual* strategies for mental operations with rational numbers. It is possible, of course, that students demonstrate conceptual understanding of a process they use, which is described as a mixed strategy.

Yang and colleagues have named subjects' strategies as *number-sense* and *rule-based* (Yang, 2003, 2005, 2007; Yang et al., 2009). Their criterion for distinguishing a strategy as based on number sense was whether one or more components of number sense are evident in the solution process (Yang, 2003, 2005, 2007). In this study, we adopt the terms conceptual and procedural strategies.

Some examples of conceptual strategies in this study are as follows:

1. *Conversion between fractions, percentages, decimals or integers* before operating. For example, for the addition of fractions, students convert them to decimals: $\frac{1}{2}+\frac{3}{4}=0.5+0.75=1.25$. In operation 1-0.25, students convert the numbers to integers: 100-25=75 so 0.75.

2. *Schematic representation of fractions* used a mental picture. For example, in subtraction 1- $\frac{1}{4}$, a student says 'I see 1 as an entire pizza or a clock with four quarters. I remove the $\frac{1}{4}$ and $\frac{3}{4}$ is left'.

3. *Benchmarking*. Students use benchmark numbers like $\frac{1}{2}$, 0.5, 1, 1%, 10% and 50%.

For example, to find 75% of 200 students thought that 100% of 200 are 200, then they have found 50% of 200 are 100 and then 25% of 200 are 50. So 75% (25% + 50%) of 200 is 150 (50 + 100).

4. *Change operation*. Here students change the proposed operation with another. For example, in 1- 0.25 one student responds "how much do I need to add at 0.25 to make 1? I want 0.75, so 1- 0.25 =0.75". It is the strategy '*Subtraction by addition (SA)*'. In multiplication 0.1x45, students said that the multiplication by 0.1 is essentially division by 10, so 45 by 10 makes 4.5.

Some examples of procedural strategies of this study are as follows:

1. *Rule of operation*. For example, the rule of addition of fractions without a common denominator: $\frac{1}{2}+\frac{3}{4}=\frac{\overset{2}{1}}{2}+\frac{\overset{1}{3}}{4}=\frac{2}{4}+\frac{3}{4}=\frac{5}{4}$. In 0.1x45 some students used the vertical algorithm of multiplication while some students used the memorized rule where multiplying by 0.1 the decimal point moves one position to the left.

RESEARCH IN MENTAL CALCULATIONS WITH RATIONAL NUMBERS

Some researchers investigated student performance on mental calculations with rational numbers (Callingham & Mclntosh, 2002; Reys, Reys, & Hope, 1993; Reys, Reys, Nohda, & Emori 1995), the kind of errors in mental calculations with rational numbers (Mclntosh, 2002, 2006), the kind of strategies used in mental calculations with rational numbers (Caney & Watson, 2003; Lemonidis & Kaiafa 2014). Finally, some researchers investigated the flexibility to move between fractions and decimals in mental calculations (Pagni, 2004; Reys, Reys, Nohda, & Emori, 1995; Sweeney & Quinn, 2000).

Reys, Reys and Hope (1993) investigated student performance in 2^{nd}, 5^{th} and 7^{th} grades in mental calculations and what types of exercises they prefer to do mentally or using paper and pencil or even a calculator. Questions for 7^{th} grade pupils also included operations with rational numbers. Very low rates of correct responses to the implementation of mental operations were observed. In order to find the result of the operation 10% of 750, only 16% of the students preferred the mental calculation, while for the sum of $\frac{1}{2}+\frac{3}{4}$ the percentage that would be used by mental calculations is significantly higher and reaches 38%.

Reys, Reys, Nohda, and Emori (1995) have recorded a wide range of performance in mental calculations with integers, decimals, fractions and percentages in research they conducted on Japanese students in 2^{nd}, 4^{th}, 6^{th} and 8^{th} grades. As is expected, exercises with decimal, fractions and percentages were given to students of the two more advanced classes. Performance in mental calculations containing decimal numbers was higher than the calculations containing fractions, while at the same time students felt more comfortable with exercises containing decimals rather than fractions. As regards the strategies followed by students, these were limited with the most popular being the mental application of written algorithms. Also, very few students were able to use alternative strategies when asked and were unable to link decimals to fractions. Similarly, other studies (Pagni, 2004; Sweeney & Quinn, 2000) have concluded that a common misconception among students is the idea that there are no relationships between fractions, decimal numbers and percentages. It is worth mentioning that less than 5% of fourth-grade pupils responded correctly to the sum $\frac{1}{2}+\frac{3}{4}$, when about 60% of the same students answered correctly to the sum 0.5 + 0.75.

Callingham and Mclntosh (2002) in a study conducted with the participation of 3,035 children across grades 3 to 10, developed a scale of eight levels with respect to student performance in mental calculations, with the eighth level mainly involving fractions, decimals and percentages, and particularly calculations using division. In the same year, Mclntosh (2002) recorded the percentages of correct answers from the previous study, highlighting the most common student errors. For grades 7 and 8, the results were as follows: in operations with fractions, success rates ranged from 42% to 67% except for $\frac{1}{2}+\frac{1}{3}$, where the success rate was 8%. In operations with decimals, success rates ranged from 30% to 58%, while in operations with percentages the range was from 8% to 74%. It is worth noting that 8% succeeded in finding 30% of 80, while 74% found 100% of 36. Mclntosh (2002) stressed that errors should be separated into conceptual and procedural errors (a distinction he made in his other research in 2006), pointing out that errors made by students in exercises with integers tended to be instrumental in nature, while errors made by students in exercises with fractions, decimals and percentages were conceptual in nature. Three representative problems in these categories in which students made errors that characterized them conceptual were: $1-\frac{1}{3}$, 0.3 + 0.7 and 30% of 80. Another finding was the difficulty of students, even of the older classes, to go from one symbolic representation of the rational number to another, for example, from percentage (75%) to fraction ($\frac{3}{4}$).

Caney and Watson (2003) conducted research with the participation of 24 pupils from Primary to High School (grade 3 to 10). In spite of the small number of participants, they recorded many strategies that students use to execute mental operations with fractions, decimals and percentages. They found that in the operations with decimals, the strategies used by the students were more instrumental, as opposed to the strategies they used in operations with fractions and percentages, which were more conceptual.

Lemonidis and Kaiafa (2014) conducted a survey with 5th and 6th grade Greek students using basic questions to examine whether students understood the operations with rational numbers that they perform and whether they could carry out these operations employing different strategies. The majority of students used strategies involving formal rules to perform mental operations with fractions and percentages.

THE PRESENT STUDY

The principal research questions of this study are:

• Does the type of symbolic representation of rational numbers (fractions, decimals and percentages) affect the accuracy, incidence of errors and type of strategy (conceptual or procedural) used by students in performing mental operations?

• Are students able to switch between different symbolic representations of rational numbers (fractions, decimals, percentages) when performing mental operations?

In this research, we examined a sample that consisted of Greek students in eighth grade. In the Greek programs, mental calculations and estimations are included in elementary school but not in middle school. Although Greek curricula include mental calculations with whole numbers, however, they only make a general reference and a piecemeal presentation of mental calculations with rational numbers, with emphasis on estimation procedures. Greek curricula and textbooks do not provide a specialized teaching proposal regarding mental calculations with rational numbers.

METHOD

The method of the personal interview was chosen rather than the written test for collecting the students' answers in order to be more confident in the participants' implementation of the mental approach to questions. In the research of Yang, Reys, R., and Reys, B. J. (2007) by the written test method, many candidate teachers who participated, although explicit instruction was given not to use pencil and paper, preferred their use and did the writing calculations using written algorithms. In order to completely avoid this, personal interviews were preferred.

Participants

The sample of the study is composed of 62 pupils of 8th grade, 30 girls (48.4%) from a public school belonging to an urban area of a large Greek city, Thessaloniki.

Procedure

The survey was conducted from mid-October to late November 2016, in the first months of the school year in the 8th grade class. Initially, a pilot study was applied to two students to examine the clarity and degree of difficulty of the questions. All individual oral interviews with all participants were conducted by one of the researchers to ensure the same manner of conducting the interview, as well as collecting and recording responses. Each student's examination was conducted individually with a personal interview in a designated place in the school and had as much time as he wanted.

At the beginning of each interview, the researcher provided the participants with information on how the interview would be conducted, the rules they had to follow during the interview, thus ensuring a standard examination environment for all participants (Callingham & Watson, 2004). Specifically, some of the instructions given were the following: 1. Participants were asked to solve, without the use of written calculations, any problems given to them and to explain in detail their way of thinking, and how they came to this particular answer. 2. Participants were asked to indicate all possible strategies they could think of to solve the problem. 3. Students were given A4 white paper which they could use, if they wished, for "short notes" that support memory (Lemonidis, 2015; B. Reys et al., 1993; R. Reys, 1984; R. Reys et al., 2014, 1995; Sowder, 1990), but without the ability to do operations on the paper using written algorithms.

The questions were presented to students orally and visually via PowerPoint by a computer. Oral and visual presentation, according to McIntosh et al. (1995), encourages students to explore a wider variety of strategies in relation to mental computing.

During the interview, when the researcher did not understand the participants' answers, standard clarification questions were used to help students not influence their thinking, as well as to help the researcher examine the solution approach and identify the strategies used by the participant to be able to classify them as conceptual or procedural ones. The clarifying questions/suggestions were taken from (Yang, 2005) and were as the following: 1. Please make your answer more specific. 2. Please tell me why. 3. Please tell me how you did it. 4. Can you find another way?

Each student's personal interview lasted from 11 to 36 minutes with an average of 20 minutes.

The interview process was as the following:

The first slide of the PowerPoint contained the research information and instructions that the participant was looking at and was simultaneously read aloud by the researcher. After explaining how the research was conducted, the student, at the request of the researcher, proceeded to the next slide containing the first question in response to the researcher reading the content out loud. By that moment the measurement of the time to answer the question began. The participant had as much time as he wanted to answer the question. The response time of the student's first response was recorded by the researcher at the specific area of the protocol.

The student was then asked to describe in detail the way of thinking, which led him to his answer, while simultaneously recording it by the researcher in the corresponding space on the standard record sheet. After completing and recording the student's first way of thinking, the researcher was asked if he could think of a second way to solve the same problem. If the student gave a second answer this would be recorded in the specific area of the protocol, followed by a question about whether he could think of a third way to solve the same problem and record it by the researcher. This process was followed until the student said that he could not think of another way to solve the same problem and then the same procedure was repeated in the next problem. Overall, for each problem, only the response time of the student's first answer was recorded, the strategy of solving the first answer was recorded, and then all the different strategies that could be thought of for that problem were recorded.

Tasks

For this survey, an observation protocol was used consisting of 12 operations' problem with fractions, decimals and percentages. There were 4 problems with fractions, decimals and percentages for the four basic operations (addition, subtraction, multiplication and division). In particular, it contained the following problems:

Parallel problems in fractions and decimals were given in each basic operation, i.e. problems with the same numbers in each operation presented in different symbolic representations, e.g. $1 - \frac{1}{4} =$ and $1 - 0.25 =$. The

Table 1. 12 problems in which students were examined

Fractions		Decimals		Percentages	
F1	$1 - \frac{1}{4} =$	D1	1 – 0.25 =	P1	25% of 80
F2	$\frac{1}{2} + \frac{3}{4} =$	D2	0.5 + 0.75 =	P2	75% of 200
F3	$\frac{1}{10} \cdot 45 =$	D3	0.1 · 45 =	P3	10% of 45
F4	$\frac{1}{2} : \frac{1}{4} =$	D4	0.5 : 0.25 =	P4	50% of 24

Table 2. Frequencies and success rates, for the first answer of students for each problem

Fractions		Decimals		Percentages	
Problem	Success	Problem	Success	Problem	Success
F1: $1 - \frac{1}{4}$	38 61.3%	**D1:** 1-0.25	57 91.9%	**P1:** 25% of 80	31 50.0%
F2: $\frac{1}{2} + \frac{3}{4}$	48 77.4%	**D2:** 0.5+0.75	37 59.7%	**P2:** 75% of 200	35 56.5%
F3: $\frac{1}{10} \cdot 45$	55 88.7%	**D3:** 0.1 · 45	38 61.3%	**P3:** 10% of 45	34 54.8%
F4: $\frac{1}{2} : \frac{1}{4}$	48 77.4%	**D4:** 0.5 : 0.25	40 64.5%	**P4:** 50% of 24	50 80.6%
N=62 Mean	76.2%	Mean	69.4%	Mean	60.5%

choice to give the same numbers to each fraction and decimals was done with the rationale that it can be compared to the strategies that the students would choose depending on the symbolic representation of the numbers, without the added difficulty of a different number. For the percentages, one of the two numbers in the fractions and decimals was used to provide a "parallelism" with the corresponding fractions and decimals (e.g. 25% of 80). With the percentages, it is not possible to find two numbers equal to the corresponding fractions and decimal numbers to make all operations 'parallel'. For this reason, we chose percentages problems with the ending number unknown. Only in the third row of **Table 1** are the numbers of problems ($\frac{1}{10}$ · 45 =, 0.1 · 45 =, 10% of 45) totally parallel in all three problems where the same number is used (45) and the rational number changes each time, respectively.

The parallel problems in order to be able to compare each other were not given to the participants consistently but in random order, although they were the same for all participants. Problems were chosen randomly to prevent students from correlating "parallel" problems and to shift thinking from previous problems. The problems were also given in such a way as to rotate the different representations of the rational numbers, that is, the first problem from the fractions, the second problem from the decimal, the third problem from the percentages, and so on.

RESULTS

Accuracy

Table 2 shows the frequencies and success rates for each problem and the first answer given by the students.

In the four problems of the fractions, the success rate of the students ranges from 61.3% to 88.7%, with an average of 76.2%, while in the decimal problems from 59.7% to 91.9%, with an average of 69.4%, and the percentages from 50% and 80.6%, with an average of 60.5%. When comparing the average success rates in fractions, decimals and percentages, we find that statistically the calculations with fractional numbers do not have higher success rates than the calculations with decimals (Wilcoxon, N = 62, z = -1.854, p = 0.064, 2 - tailed). While fractional calculations have a statistically higher average success rate than percentage calculations (Wilcoxon, N = 62, z = -3.866, p <0.000, 2-tailed) and calculations with decimals have higher average success rates than calculations with percentages (Wilcoxon, N = 62, z = -2,037, p = 0.042, 2-tailed).

Thus, in our research sample, it appears that on average, mental operations with fractions and decimals are statistically easier than the corresponding operations with percentages. While between the operations of the fractions and the decimal there does not appear a statistically significant difference in difficulty.

Table 3. Frequencies and corresponding rates of use of conceptual (CON) and procedural strategies (PR) (only for the first successful strategy)

Fractions				Decimals				Percentages			
N=62		CON	PR			CON	PR			CON	PR
$1-\frac{1}{4}$	n	18	20	1-0,25	n	43	14	25% of 80	n	18	13
	%	29	32.2		%	69.3	22.5		%	29	20.9
$\frac{1}{2}+\frac{3}{4}$	n	3	45	0,5+0,75	n	22	15	75% of 200	n	19	16
	%	4.8	72.5		%	35.4	24.1		%	30.6	25.8
$\frac{1}{10}\cdot 45$	n	7	48	$0{,}1\cdot 45$	n	18	20	10% of 45	n	4	30
	%	11.2	77.4		%	29	32.2		%	6.4	48.3
$\frac{1}{2}:\frac{1}{4}$	n	2	46	0,5 : 0,25	n	34	6	50% of 24	n	44	6
	%	3.2	74.1		%	54.8	9.6		%	70.9	9.6
Mean	%	12.1	64.1	Mean	%	47.1	22.1	Mean	%	34.2	26.2
regardless of success	%	15,9	84,1		%	68,1	31,9		%	56,6	43,4

Errors

Student errors were separated into conceptual errors and errors made when using procedural strategies. An error due to conceptual error is because the student has not sufficiently understood the nature of the numbers, while the error in the use of rules (procedural error) is the one in which the student makes an error in execution of the algorithmic rule (Mclntosh, 2002, 2006). For example, in operation 0.5 + 0.75, 25 (40.3%) of the 29 total errors were conceptual errors. The most common error was the response 0.80 because 5 + 75 = 80. This shows that these students have not understood that 5 represents five tenths while 75 represent seventy-five hundredths. These difficulties are due to the correct return of place value to the digits of a decimal number. In operation 1- $\frac{1}{4}$, 21 (33.8%) of the 24 total errors were procedural errors, and in particular the misunderstanding of the rule of fractions subtraction. Some students removed the numerator of the fraction from 1 and found $\frac{0}{4}=0$, while others removed the denominator from 1 and found $\frac{0}{3}$ or $\frac{0}{-3}$.

Student errors using rules came to 142 (68.9% of total errors), while there were 64 (31.1% of total errors) conceptual errors. We note that there were more than twice as many uses of rules errors as conceptual errors.

The Strategy used in Relation to the Symbolic Representation of the Rational Numbers

The different strategies used by students in mental calculations of fractions, decimals and percentages were grouped into two groups, conceptual and procedural strategies. For example, in the first problem of fractions (1- $\frac{1}{4}$), students used a total of five different strategies, four of which were conceptual and only one procedural strategy. The four conceptual strategies used by the students were as follows: a) 19 students (39.6%) *converted the fraction to decimal*, i.e. $\frac{1}{4}$ to 0.25, and then did the subtraction and they found 0.75 (1- $\frac{1}{4}$ = 1-0.25 = 0.75). This was also the most popular conceptual strategy used by students. b) 10 students (16.1%) *converted the whole number to fraction* ie the 1 to $\frac{4}{4}$ and then subtracted (1- $\frac{1}{4}=\frac{4}{4}-\frac{1}{4}=\frac{3}{4}$). c) Only one student (1.6%) made *a mental schematic representation* by dividing 1 in four equal pieces from which he removed one of them and left him three of the four pieces, finding the result $\frac{3}{4}$. d) Also, one student (1.6%) calculated numerically with fractions, said that $\frac{1}{4}$ to become whole, ie 1, wants still $\frac{3}{4}$.

25 students (32.3%) used the procedural strategy. Thus, they used *the addition rule of the fractions* by converting the fractions with the same denominator $1-\frac{1}{4}=\frac{1}{1}-\frac{1}{4}=\frac{\overset{4}{1}}{1}-\frac{\overset{1}{1}}{4}=\frac{4}{4}-\frac{1}{4}=\frac{3}{4}$. Finally, three students (4.8%) gave no answer.

Table 3 lists the frequencies and corresponding rates of the conceptual and procedural strategies used by students in each problem. Only the strategy used by the students in their first successful answer to each problem has been taken into account in the following results. On the contrary, the last row of the table presents the strategies that students used in their first answer to each problem, regardless of whether the answer was right or wrong.

According to the data in **Table 3**, we can observe that in the operations with fractions the majority of students use procedural strategies (64.1% vs. 12.1%), while in decimal operations the majority of students use

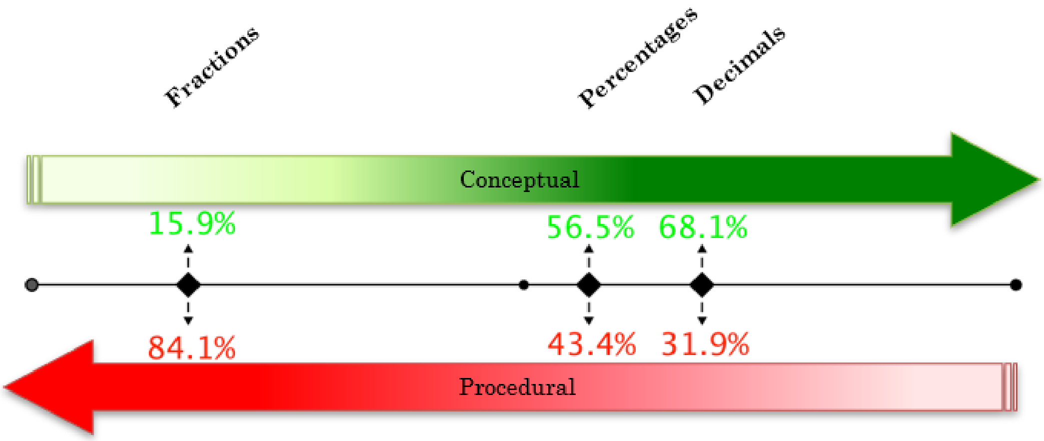

Figure 1. Strategy used with fractions, decimals and percentages

conceptual strategies (22.1% vs. 47.1%). In operations with percentages there is a statistically significant difference in the average of using procedural or conceptual strategies (34.2% vs 26.2%) ($z=1.95$, $p<0{,}05$).

The rates of use of the two groups of strategies vary according to the numbers in each type of operation. For example, the number 50% in the operation 50% of 24 leads the majority of pupils to use conceptual strategies (70.9% vs 9.6%).

There is an impressive change in the use rates of the procedural and conceptual strategies in the three parallel operations $\frac{1}{10}$ .45, 0.1.45 and 10% of 45, where the same number appears as a fraction, decimal and percentage and is combined with 45. When we have the fraction $\frac{1}{10}$ and the percentage 10% in the operations $\frac{1}{10}$. 45 and 10% of 45, the vast majority (77.4% and 48.3% respectively) of students use procedural strategies. While with the decimal 0.1 in the operation 0.1. 45 the rates of use of procedural and conceptual strategies are almost equal (29% and 32.2%).

Consequently, we can say that the vast majority of students in operations with fractions use procedural strategies rather than the operations with decimal numbers; students use the conceptual strategies more. Procedural strategies are also more used in operations with percentages. Also, students use more conceptual strategies in percentages operations than in fractions operations. By placing the different symbolic representations of rational numbers on one axis in terms of the kind of strategy the students use, we have the schematic representation shown in **Figure 1**. The middle of the axis represents 50% and the two ends 100%. The green and the right direction represent the conceptual strategies while the red and the left direction represent the procedural strategies. The percentages of strategy use are from the bottom line of **Table 3**. For example, in the fractions 84.1% is the percentage of procedural strategies while 15.9% is the percentage of conceptual strategies.

Ability to Convert in Different Symbolic Representations

Of the total 744 (12x62) problems posed to all students, in 224 (30.11%) of these students did not use any strategy successfully, while in another 308 (41.4%) the students used only one strategy correctly. Cumulatively, in 532 problems, which make up 71.51% of the problems, students either did not use any strategy or used a single strategy successfully. In 177 (23.79%) problems, the students used 2 different strategies successfully, while the number of problems that used 3 or 4 different strategies successfully was just 35, representing 4.71% of the problems.

The students of the sample had a small repertoire of strategies in mental calculations with rational numbers, with most students using only one strategy (41.4%), fewer (23.79%) two strategies, while very few (4.71%) can use three strategies and more.

The ability to convert and transition from one symbolic representation of the rational numbers to another is part of the student's flexibility. We will attempt to demonstrate how this ability is shown in this student sample.

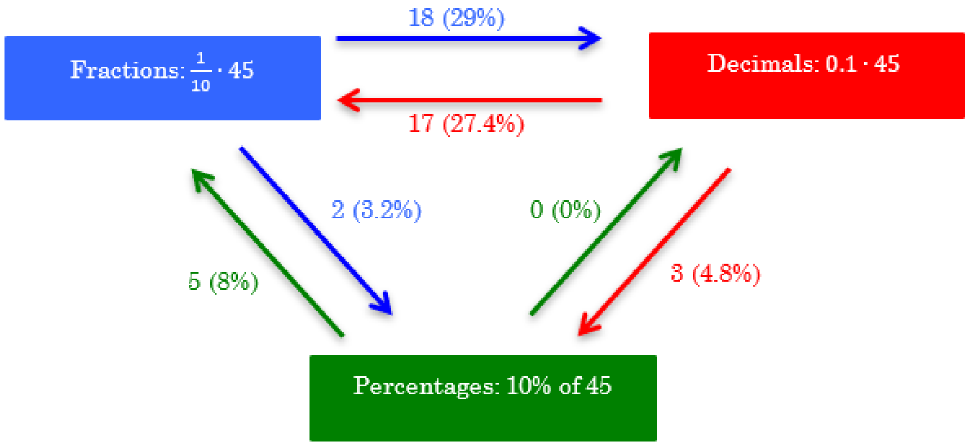

Figure 2. The number of students who were able to move between the different symbolic representations of the rational number in problems $\frac{1}{10}$. 45, 0.1. 45 and 10% of 45

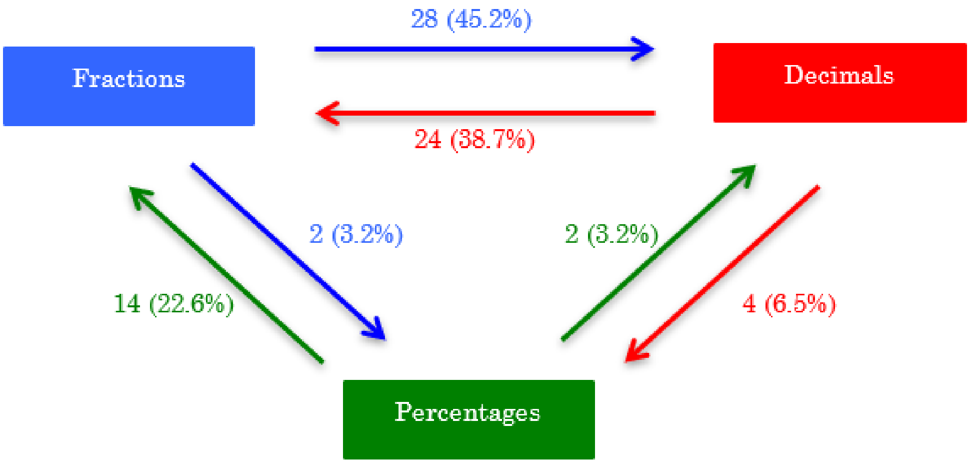

Figure 3. The number of students who were able to switch between the different symbolic representations of the rational numbers in the set of problems asked

As we can see in **Figure 2**, in parallel questions ($\frac{1}{10}$. 45, 0.1. 45, 10% of 45), 18 students (29%) converted fractions into decimals, while 17 students (27.4%) converted decimals into fractions. No students converted the percentages into decimals, while only 3 students (4.8%) converted decimals into percentages. 5 students (8%) converted the percentages into fractions, while 2 pupils (3.2%) used as a strategy the conversion of the fractions into percentages.

Figure 3 shows the number of students who were able to switch between the different symbolic representations of the rational numbers in all the problems asked.

Shifts from fractions to decimals and vice versa were made by fewer than half the students. In particular, 28 students (45.2%) were able to convert fractions into decimals even only in one strategy on any of the problem asked. The conversion from decimal to fractional was achieved by 24 students (38.7%).

The conversion of the percentages into fractions was made by 14 pupils (22.6%), while only 2 pupils (3.2%) did the conversion from fractions into percentages.

There were few students who converted decimal into percentages and vice versa. Only 4 students (6.5%) converted the decimals into percentages, even only in one strategy on any of the problems asked and 2 students (3.2%) made the conversion from percentages to decimal.

From the above data, we can observe that pupils do not associate and cannot link the different symbolic representations of the rational numbers with each other. Fewer than half of the students can alternate fractions with decimals. Less than a quarter of pupils can convert percentages into fractions, with the rate of students performing the reverse rotation, ie fractions in percentages, being extremely small (3.2%).

There are also small rates (close to 5%) of students who alternate decimal with percentages and vice versa. The ability to manipulate and to switch between the different symbolic representations of the rational numbers is very small especially when the switch involves percentages. In conclusion, we can say that the students of the sample cannot be considered flexible as regards the transition from one symbolic representation of the rational numbers to the other.

CONCLUSIONS AND DISCUSSION

Performance in Mental Calculations with Rational Numbers of Different Symbolic Representation

According to the results presented above regarding student performance, two out of three responded correctly to the problems asked with success rates ranging from 50% to 92%. Performance in mental operations with fractions does not produce a statistical difference in difficulty from the corresponding operations with decimals. There is research (Callingham and McIntosh, 2001; Callingham and Watson, 2004) that show that students find better performing in operations with fractions rather than with decimals. Other investigations (eg DeWolf, Grounds, Bassok, & Holyoak, 2014; Iuculano & Butterworth, 2011; Reys, R., Reys, BJ, Nohda, & Emori, 1995) have reached the opposite conclusion, that operations with decimals are easier for students than operations with fractions. In the present study, the last conclusion is only confirmed in a pair of parallel questions $1-\frac{1}{4}$ and 1- 0.25.

In the parallel problems that appeared in the same operation involving different symbolic representations of rational numbers (fractions and decimals), there were great differences in student success rates. Very few students seem to be able to make the connection between the different symbolic representations of rational numbers by making the transition from one representation to another, confirming the conclusion of many researchers (Gay & Aichele, 1997; Hiebert & Hiebert, 1984; Lemonidis & Kaiafa 2014; Mclntosh, 2006; Pagni, 2004; R. Reys et al., 1995; R. Reys & Yang, 1998; Sweeney & Quinn, 2000). If the students were aware of the connection and the transition from one symbolic representation of the rational to the other and were using it correctly, then the success rates in the parallel problems asked should have been very close.

Pupils' performance in operations with percentages was worse than their performance in operations with other representations of rational numbers (fractions and decimals), the only exception being the operation 50% of 24. The large success rate in operation 50% of 24 is due to the fact that the 50% is very familiar to them and used quite often in everyday life, an interpretation given by Gay and Aichele (1997). A very large number of students, who in some problems exceeded 25%, did not give any answer. These students stated that they cannot compute with percentages because they simply do not understand them. Lembke and Reys (1994) have expressed the same view that percentages are one of the most difficult subjects to understand not only by students but also by adults.

Thus, there were large fluctuations in the performance of pupils, which according to the results, appear to depend on three factors: the type of operation, the type of numerical data of the operation's terms and the type of symbolic representation of the operation's numbers.

Strategies Used in Different Symbolic Representation of Rational Numbers

With regard to the type of strategy that students choose for mental calculations with rational numbers, they have been observed to choose procedural strategies when dealing with operations with fractions, as opposed to decimal operations in which they choose conceptual strategies. Lemonidis, Tsakiridou and Meliopoulou (2018) also reached the same conclusion in their research in 70 in-service teachers. Lemonidis and Kaiafa (2014) also found a low number-sense in operations with fractions in their research among 5th and 6th grade students with most students making use of procedural strategies. However, these findings contradict the Caney and Watson (2003) study in 24 students in Australia (grades 3 to 10) and found that most students use procedural strategies in operations with decimal numbers instead of the operations with fractions and percentages that use conceptual strategies. Also, Yang's (2005) research of 6th grade students in Taiwan found results opposite to those of the present study, with students in decimal operations tending to apply rules-based strategies. Carvalho and da Ponte (2013) in their research of 6th grade students in Portugal found the same results as the present study - as shown the beginning of the study (before the intervention was implemented) - in operations with fractions, pupils used mainly procedural strategies. It is worth noting that following an intervention, students used more and more conceptual strategies, which may indicate how pupils could learn based on the number sense.

Caney and Watson (2003) in their research of Australian students (grades 3 to 10) found that pupils in operations with percentage use more conceptual strategies while Lemonidis and Kaiafa (2014) in their research into Greek fifth and sixth grade pupils drew the opposite conclusion, that is, that students use more procedural strategies. In the present study, there is no clear picture of the type of strategy students follow when calculating with percentages, so that no conclusion from the above research can be confirmed. The choice of strategy seems to be determined by the type of numerical data involved in operations with those most familiar in everyday life (for example 50%) being approached using conceptual strategies, while, with numerical data that are less familiar, students approach them by using procedural strategies.

The conclusion that we can thus draw from the research into the kind of strategy that students use in mental calculations with rational numbers is that they use more procedural strategies when they have to make mental calculations with fractions while using conceptual strategies, when the operations contain decimals. For operations containing percentages it is not clear what kind of strategies they use, with their choice being greatly influenced by the numbers involved in the operation. Compared with the types of symbolic representation of the rational numbers, students use conceptual strategies more in operations containing decimals, immediately after operations in percentages and less in operations with fractions.

It would appear that students seem to understand decimals because there is a similarity in their appearance with integers (Caney & Watson, 2003; DeWolf et al., 2014), and this understanding encourages students to use conceptual strategies. Conversely, students who seem not to understand fractions, use more strategies based on algorithmic rules - without knowing whether the rules actually work (Hasemann, 1981) - by linking the conceptual understanding of the fraction to the processes not being necessary for their implementation (Nunes & Bryant, 2009).

The Ability to Alternate on Different Symbolic Representations of Rational Numbers

Another question raised in this study was whether students are able in switching between different symbolic representations of rational numbers (fractions, decimals, percentages) when performing mental calculations. From the results it is clear that students have not developed a connection between the different representations of the rational number in order that they can move from one representation of the rational to another, a conclusion that is in full agreement with the conclusions of other researchers (Gay & Aichele, 1997; Hiebert & Hiebert, 1984; Lemonidis & Kaiafa, 2014; Mclntosh, 2006; R. Reys et al., 1995; R. Reys & Yang, 1998; Sweeney & Quinn, 2000). It is also clear that the repertoire of the strategies used by students is very limited, confirming the conclusions of other researchers (Lemonidis & Kaiafa, 2014; R. Reys et al., 1995). Most students have used at most two strategies, with few students using at least three strategies. It can therefore be concluded that students cannot be considered flexible in switching between different symbolic representations of rational numbers when performing mental operations.

Implications for Instruction

One of the main findings of this research and of other international studies presented above is that secondary school students find it difficult to see the relationship and therefore move from one symbolic representation to another when calculating mentally with rational numbers as benchmark. It is therefore necessary to place emphasis in instruction to the relationship between and the passage from one symbolic representation to the other at least for the rational numbers as benchmark. It is interesting to observe that in modern research proposals for the teaching of fractions (e.g. Fazio & Siegler, 2011) while special reference is made to the necessity of using the estimation in teaching fractions, there is no reference to the necessity of mental calculations in operations with fractions nor either to the connection of the different symbolic representations of the rational numbers as benchmark.

We believe that the switching capability between the three symbolic representations of rational numbers as benchmark could be one of the key elements that determine number sense in middle school students. This ability is obviously essential and indispensable in the everyday life of citizens and is therefore a key feature of adult numeracy.

Limitations - Extensions of the Study

Our research was conducted on a limited number of students in one specific education system. It would be interesting to carry out a similar survey on a larger number of students from different education systems with different teaching approaches to rational numbers.

Within the framework of adult numeracy, it would also be interesting to look at the behaviors of adults or special professional groups on this topic.

Disclosure statement

No potential conflict of interest was reported by the authors.

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