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Progressive mathematics of functions in secondary school students using a free-fall activity

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ABSTRACT

This article aims to analyze the process of the progressive mathematization of the concept of the function by secondary school students through the resolution of a free fall situation. A free-fall mathematical situation was designed and experimented using the Tracker software to obtain movement data. Worksheets and conversations were collected between students and the teacher, and experiments and class discussions were filmed. The analysis consisted of identifying the development from informal strategies associated with the context of the problem to formal procedures. The results show that the process brings meanings to mathematical ideas such as variation, variables, and the relationship between variables, allows coordination and transition between representations and contributes to the learning of functions as a representation of the variation of physical phenomena.

Keywords: progressive mathematization, functions, free fall, software

INTRODUCTION

The function is one of the most relevant concepts in mathematics, present in much of higher mathematics and various areas of knowledge (Eisenberg, 2002). Teaching of functions generally begins in secondary school, where students have first contact with this concept. However, the complexity of mathematical notation, the restricted teaching contexts, and the limited set of tasks chosen by teachers make this topic particularly complex to teach and learn, making it a problem area (Sajka, 2003).

Function teaching in secondary school often focuses on mastering techniques and procedures, either as a process for representing a class of ordered pairs (Niss, 2014) or as an input-output machine (Doorman et al., 2012). This approach is insufficient when students are faced with situations that require a global interpretation of the function, a qualitative understanding of the variables (Duval, 2006), and skills such as analyzing the behavior of graphs or translating information from a graph into an algebraic expression or table of values (Best & Bikner-Ahsbahs, 2017).

With an algorithmic approach, students often need help distinguishing the relationship between the graph and the function (Eisenberg, 2002) and rarely recognize the fundamental ideas of relationship and dependence between variables underlying the concept of function (Falcade et al., 2007). The understanding of functions goes beyond manipulating algebraic formulae and mastering procedures (Breidenbach et al., 1992). Developing skills in representing data and reading and deciphering information contained in graphs is crucial.

Mathematical knowledge acquired through an algorithmic approach is often quickly forgotten as it is not linked to previous experiences (Jonsson et al., 2014). This scenario has motivated the exploration of new methodologies, where mathematical concepts are constructed from students' experiences and experiences in solving situations and problems, as in mathematical modeling (Best & Bikner-Ahsbahs, 2017). In this approach, mathematical concepts are used as tools to study problems, situations, and phenomena rather than as a goal in themselves. Interaction with everyday life offers a favorable setting for mathematics to come alive and gain meaning. It is here, in this fusion of abstract concepts with tangible experiences, that mathematical modeling becomes an essential pedagogical tool. Mathematical modeling is an activity that seeks to solve real-life problems by constructing simplified mathematical representations in order to understand, reflect, analyze conditions, and make decisions, among other objectives. Students accept the challenge of manipulating variables and establishing structured relationships during this activity and progressively mathematize the situation. Research such as that of Arzarello and Robutti (2004) and Ortega et al. (2019) have shown that experimental activities involving the analysis of movement allow reflection on physical experience and favor the construction and interpretation of graphs by identifying variables, numerical tables, and relating graphs and functions to real-world situations. In addition, it has been observed that students establish connections by translating their movements into graphs and vice versa.

As Gravemeijer (2007) points out, teaching mathematics through modeling is based on the use of problems in meaningful contexts that provide opportunities to formulate mathematical representations (graphs, figures, and diagrams) and that, at the same time, promote specific reasoning and evoke mathematical tools and concepts. Freudenthal (2002) argues that mathematics results from human activity and, therefore, teaching based on modeling encourages guided reinvention.

In realistic mathematics education (RME), students undergo a process of mathematization similar to that recorded in the historical development of mathematics. That is, they start from their own informal strategies, ideas, and intuitions linked to the conditions and characteristics of the context, progressively moving towards increasingly abstract and formal formulations. This teaching perspective enables students to connect mathematics to the real world and to develop critical thinking and problem-solving skills.

Aim & Research Questions

Teaching the concept of function is often limited to procedures and techniques, which does not allow for a deep understanding of the link between the graph and the function or the relationship between variables. Mathematical modeling offers a pedagogical alternative, suggesting using real-world problems to develop mathematical representations, fostering a deeper understanding of everyday life scenarios. Experimental activities like motion analysis enhance the interpretation and connection between mathematical concepts and real experiences. This study aims to design and implement a didactic activity to study the function through modeling the free fall of a ball, using the Tracker software as support. The activity was designed following Freudenthal (2002) realistic mathematization approach and inspired by the works of Arzarello and Robutti (2004) and Ortega et al. (2019), which focused on experimental activities and the study of functions. The purpose of this study is to investigate the progressive mathematization process of students through an analytical framework of mathematical activity that examines the learning process of the function concept in the context of experimental activities. The research question is:

How does the incorporation of mathematical modeling from a realistic mathematics perspective influence the development and understanding of the function concept, particularly in the identification and relationship of variables, and how is this progressive mathematization manifested in students' learning?

THEORETICAL FRAMEWORK

This research is grounded in the theoretical framework of RME, an instructional mathematics theory emphasizing the importance of rich and "realistic" situations in learning. These situations are a starting point for developing concepts, tools, and mathematical procedures, and a context in which students can subsequently apply their mathematical knowledge. Although the application in "real-world" situations is valued, the term "realistic" in RME has a broader meaning, referring to problems that students can imagine or visualize in their minds, whether from the real world, the fantasy world, or the mathematical world itself (Van Den Heuvel-Panhuizen & Drijvers, 2020). In RME theory, learning emerges from the student's mathematical activity and not from the memorization of prefabricated axioms, so the challenge in teaching is to provide appropriate contexts in which students experience mathematics as a human activity (Van Den Heuvel-Panhuizen, 2020), make discoveries (Scherer, 2020), and articulate their ideas and conceptions to reinvent mathematics (Wittmann, 2020).

Kaur et al. (2020) identify the basic principles of teaching within RME: activity principle, where students are active participants in the learning process; level principle, where mathematical learning presents different levels of understanding ranging from informal solutions related to the context to knowledge linked to concepts and strategies; interweaving principle, which advocates integrating mathematical domains such as arithmetic and geometry into situations or problems rather than approaching them as isolated content; principle of interactivity, according to which learning mathematics is a social activity involving collaboration, interaction and reflection on diverse mathematical productions; principle of guidance, where teachers play a proactive role in students' learning, fostering the development of models that support mathematics learning and curricula with coherent long-term teaching and learning trajectories; and principle of reality, which states that the study of mathematics starts from problem situations in realizable or imaginable contexts as starting points for mathematics learning.

According to Selter and Walter (2020), learning is a consequence of mathematization. This process leads to the development of formal thinking based on several principles. Mathematization is a construction and reconstruction activity involving the use of real solvable problems by learners, where the term "realistic" does not necessarily imply that a situation is actual. Moreover, it is based on reflection on one's own and others' thought processes; the teaching seeks to stimulate students to review and reflect on their learning process. Mathematics is embedded in a socio-cultural context, taking advantage of opportunities for communication and cooperation. It also promotes the construction of knowledge and skills for a structured mathematical entity, and the teaching seeks to weave together different conceptual elements to enable learning. Mathematics is a long-term process that evolves from the concrete to the abstract. Through teaching, educators aim to elevate students from using context-specific, informal techniques to employing structured, formal ones.

Van Den Heuvel-Panhuizen (2020) stresses the importance of students having opportunities to integrate their intuitive ideas, rudimentary strategies, and their own language into mathematical activity. From this, two essential important implications for the design of activities are deduced. First, they select situations, where students actively engage in experiments, exposing their ideas, conjectures, and reasoning to their peers (Sun & He, 2020) and second, encouraging the generation of diverse mathematical representations, from drawings, traces, and diagrams to abstract representations (Kaur et al., 2020). These early mathematical

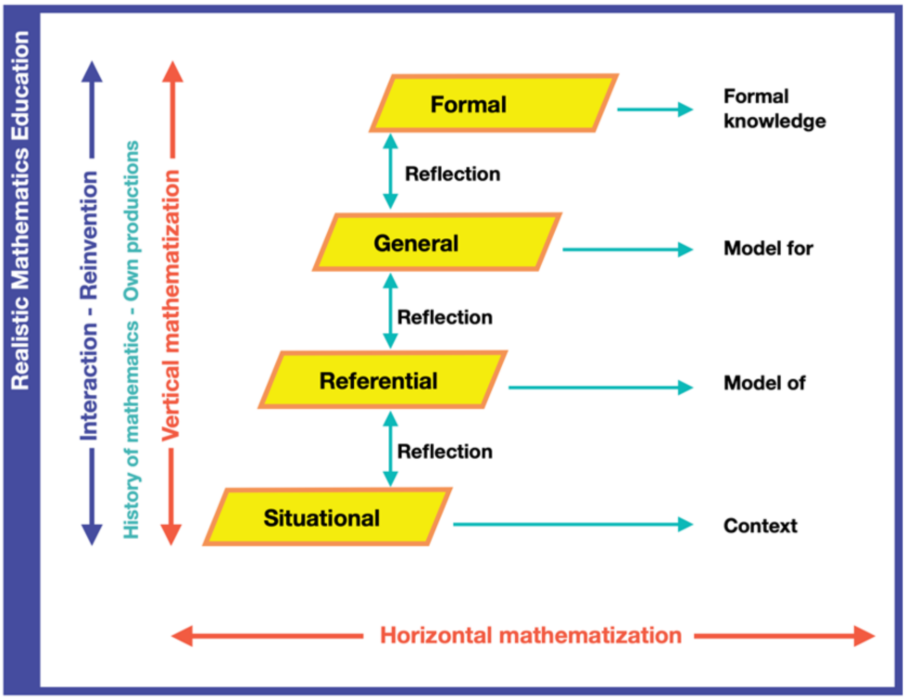

Figure 1. Description of progressive mathematics process (Bressan et al., 2016, p. 7)

manifestations of students represent the starting point for the development of formal thinking through organizing, structuring ideas, producing models, diagrams, and symbols, i.e., moving towards progressive mathematization.

According to Van Den Heuvel-Panhuizen (2003), Gravemeijer (1994) extended and refined the ideas presented by Streefland (1990) and Freudenthal (2002) on using models in realist mathematics education. While Streefland (1990) and Freudenthal (2002) introduced the distinction between "model of," which represents an already existing reality, and "model for," which guides the creation or understanding of a new reality, Gravemeijer (1994) connected this transition between models to the general process of progressive mathematization. Gravemeijer (1994) argued that as students deepen their mathematical understanding, how they use and understand models evolves. Gravemeijer (1994) introduced an additional division within the intermediate level of mathematical understanding to provide a more detailed structure to this process. He proposed that between the situational level, related to tangible contexts, and the formal level, related to abstract concepts, there are two additional levels: the referential level and the general level. The referential level bridges concrete situations and abstract mathematical concepts, using references and analogies from the real world. On the other hand, the general level represents a more advanced stage in which ideas are less concrete and more generalized but do not yet reach the degree of abstraction of the formal level.

The situational level is characterized by strategies and informal knowledge linked to the context, common sense, schematization, discovering relationships, and regularities. The referential level involves the emergence of the first models, such as representations, graphs, strategies, and notations, to schematize the problem (model of). At the general level, the models found are generalized by reflecting on strategies, moving away from the context, and recognizing that they are applicable to other problems (model for). At the formal level, concepts, procedures, and mathematical notation are understood and acted upon to interpret and resolve other problems (formal knowledge).

Figure 1 show the relationship between the four levels of understanding and their features, according to Bressan et al. (2016). The information in **Figure 1** describes the two types of mathematics in RME. Horizontal mathematization is the process that connects the "world of life" with the "world of symbols." It is the translation of real-world problems or situations into a mathematical format. This translation involves taking a real scenario, problem, or context and transforming it into a mathematical problem that can be worked and solved. Such activity is not passive; it involves active reinvention on the part of the learner. Through reinvention, learners, based on their own productions, make sense of mathematics. These learners' productions are essential as they bridge their prior knowledge and the formal knowledge they are expected to acquire.

Moreover, interaction plays a fundamental role in this process. Students refine, question, and expand their ideas by interacting with their peers and teachers. These collaborative discussions are essential to enrich the horizontal mathematics process. Through this interaction, students also begin to understand the relevance and applicability of context in mathematics, as they rely not only on hypothetical situations but on real scenarios that have meaning and relevance in their daily lives.

Vertical mathematization, on the other hand, deals with the work done within the mathematical system. Once the real-world problem has been translated into mathematical terms through horizontal mathematization, vertical mathematization is about manipulating, solving, and exploring that problem within the language and rules of mathematics. This is, where formal knowledge becomes essential. Students must apply formal mathematical rules and theorems to solve the mathematized problem. However, it is crucial that this formal knowledge is not taught in isolation but is deeply rooted in students' context and prior experiences for mathematics to be meaningful and helpful in solving real-world problems.

Fischbein (2002) argues that mathematical concepts have an intuitive charge fundamental to reasoning. The challenge in teaching is to get the student to productively control their intuitive ideas so they are accurate during their mathematical activity. One way to address this challenge is to create teaching situations that help students become aware of such conflicts.

According to Fischbein (2002), mathematical concepts have an intuitive charge fundamental to reasoning. The task in teaching is to ensure that the student can productively control his intuitive ideas to avoid being misled by them during his mathematical activity. One way to meet this challenge is to create teaching situations that help students become aware of such conflicts.

About the Functions & Progressive Mathematization Process

Throughout history, the notion of function has undergone significant changes. Initially, it was closely linked to physics problems, with an implicit focus on the concept of motion. However, this idea was subjected to a process of formalization and axiomatization that led to a substantial change in the definition of functions, shifting them from a motion-based interpretation to a more algebraic modern definition (Falcade et al., 2007). In this evolution, it went from being recognized as an equation with Euler (a function y of x is an analytical expression denoted as $f(x)$, which expresses y in terms of x) to an arbitrary correspondence in a numerical interval with Dirichlet (y is a function of x if to each x is associated a value of y called $f(x)$), to a correspondence between any pair of sets not necessarily numerical in the framework of set theory with Bourbaki (a function from a set A to a set B is a subset C of the Cartesian product $A×B$ with the property that for every x in A there is exactly one y in B such that (x, y) is in C). Finally, to be defined as generalized distributions or functions (Kjeldsen & Lützen, 2015).

Although the more modern definitions, such as Bourbaki sets, are often presented in teaching, some researchers, including Kjeldsen and Lützen (2015) and Michelsen (2006), find this approach excessively abstract, especially for secondary education. Even Eisenberg (2002) claims it is counterintuitive and that transferring learning between different contexts and representations is difficult and only occurs in similar situations. Some strategies for studying functions, such as interaction with physics through experimentation supported by Dirichlet's definition of function, offer several didactic advantages and can significantly enrich the understanding of functions. In principle, this definition allows for a wider variety of functions, including those that cannot be expressed by algebraic formulas, making it possible to address a wide range of phenomena and relationships. In addition, it allows modeling a wide range of real phenomena that may not be continuous in the traditional sense, which is essential for many practical applications.

Studies such as Falcade et al. (2007) and Hazzan and Goldenberg (1997), state a connection between the formal static definition of function and the basic metaphor of motion. This connection is recognized in the idea that a function can be interpreted as a type of movement or variation, where one variable changes as a function of another. Thus, a function is not just a verbal label derived analytically from different static forms but has an active nature, directly observable and open to experimentation.

A qualitative analysis of a phenomenon and the relationships between the quantities that model them is key to recognizing the idea of variables and relationships. Despite their relevance and apparent simplicity, many students still need to fully understand these notions (Eisenberg, 2002), because teaching fosters procedural mastery without providing an underlying conceptual understanding (Chinnappan & Thomas, 2003). In this sense, Biehler (2005) states that teaching the concept of function should go beyond its algebraic representation; students should experience situations, where they sketch qualitative curves and emphasize the geometric aspect of functions more.

The idea of variable is fundamental in mathematics because it serves as a building block for all abstractions. It is an entity that can take different values and is therefore essential for representing and solving mathematical problems (Eisenberg, 2002). In the context of modeling, variables can be identified in various representation scenarios, both numerically and in tables, graphs, algebraic formulations, ordered pairs and so on. The challenge in didactics is to foster the recognition of this idea in a variety of contexts in order to establish an analytical formulation without neglecting the visual or intuitive basis (Eisenberg, 2002).

Next, a relationship is established between learning functions and the four levels of understanding proposed in the theory of RME. According to Eisenberg (2002), Dirichlet's definition of function provides ample opportunities for deep learning of the concept of function. These levels have been designed with that definition, emphasizing two key concepts: the idea of variables and the relationship between variables.

Situational level

Understanding of the function linked to physical problems. At this level, the function is rooted in concrete situations and linked to physics problems, precisely motion. The student is expected to schematize discover relationships and regularities in real phenomena. Understanding based on the basic metaphor of motion. The function is recognized as a variation or movement, where one variable changes as a function of another. Functions are seen as active, observable, and experimental.

Referential level

The function is seen through more defined models. In addition, visual representations of the function, such as graphs, are recognized, and situations, where qualitative curves are sketched are explored. The geometric aspect of functions and the representation of these in different scenarios, either numerically, in tables or graphs, are emphasized.

General level

Conceptual understanding. Students are introduced to the importance of going beyond algebraic representations and are encouraged to explore geometric aspects of functions and sketch qualitative curves. At this level, the function is detached from the purely physical context and g generalized to different types of representations, such as algebraic, graphical, and numerical.

Formal level

At this stage, the student approaches the function mainly from an algebraic perspective. At this level, students state, identify, and work with intrinsic properties of functions, such as determining increasing or decreasing behavior, defining and analyzing intervals of growth and decrease.

METHODOLOGY

Design of the Activity

RME emphasizes using problems or situations as a central element in teaching proposals, as they stimulate students' free and spontaneous productions (Scherer, 2020). Following this approach, a didactic activity composed of two experiments was designed. The first focuses on filling a container with water (this first experiment is not reported in this paper), while the second, the subject of reflection and analysis in this article, deals with the free fall of a ball. Although the study of this experiment from the kinematics perspective involves abstract concepts such as uniformly accelerated motion, gravity constant, and velocity, the activity focuses on the analysis of the time-position relationship, which facilitates the construction of an algebraic model and its comparison with the equations of free fall.

The didactic sequence proposes the study of a phenomenon familiar to students, which is realistic and allows the use of intuitive ideas, the formulation of mathematical representations, and the identification and analysis of initial conditions and conjectures about the phenomenon's behavior (Scherer, 2020). In analyzing the activity, students take on the role of producers of mathematics (Selter & Walter, 2020), becoming involved in both planning and obtaining, analyzing, and processing the data collected from the experiment, which can stimulate their interest and foster a positive attitude towards mathematics.

The teaching sequence is divided into four phases. In the first phase, the teacher gives a brief verbal introduction describing the situation of a free fall of a ball from a certain height. Then, the students are asked to construct a Cartesian graph on white paper, allowing them to freely choose variables and define the reference system and the shape of the graph. The aim is for students to apply their prior knowledge, ideas, and conceptions about the situation and to anticipate the shape of the graph. In the second phase, the experiment is filmed with a mobile device to make measurements and construct a data table. The purpose is to reflect on the meaning of the variables and to analyze their relationship on the basis of the data table. In the third phase, the experiment is repeated, and the Tracker software is introduced to simplify obtaining the position-time coordinates and support the qualitative analysis of the resulting graph. In the fourth phase, the resulting graph is qualitatively analyzed and compared with the initial graph, and procedures for formulating an algebraic quadratic function are explored. With the support of GeoGebra graphing software, the graph obtained is compared with the graphs derived from the free fall formulas.

Two key moments have been identified for students' internal validation of results. The first occurs when the software returns the graph of the experiment, allowing students to compare their graphs, analyze characteristics and locations in the reference system, and reflect on possible errors. The second moment occurs at the end of the activity when the three graphs are compared, and their characteristics are analyzed qualitatively and quantitatively. This confrontation fosters deep learning and critical reflection, as students can question their initial assumptions, recognize patterns and establish relationships between the different mathematical representations (Kaur et al., 2020).

RME promotes a student-centered approach to learning through relevant and contextualized problem-solving (Van Den Heuvel-Panhuizen, 2020). Designing didactic activities such as the proposed sequence seeks to develop critical thinking skills, mathematical reasoning, and effective communication among students while encouraging them to participate actively in their learning process (Sun & He, 2020).

Basic principles of RME identified by Kaur et. al (2020), are the main referents for the design of the didactic activity, as follows:

- **Activity principle:** The didactic activity promotes the active participation of the students from the beginning. By allowing them to choose variables, define reference systems, and design graphs based on their previous knowledge, students become central actors in their learning process, clearly reflecting the activity principle.
- **Level principle:** The proposed sequence goes from the initial intuitive analysis of the time-position relationship to the use of specialized software to formulate an algebraic quadratic function. The level of depth evidences the recognition of different levels of understanding in mathematical learning.
- **Principle of intertwining:** Rather than treating kinematics and mathematics as separate domains, the activity integrates both into a single experiment. This reinforces the integration of different mathematical domains and shows the practical relevance of mathematics in physics and other fields.
- **Principle of interactivity:** The activity involves student collaboration in data collection and analysis. In addition, the internal validation of the results allows for collaborative discussions and reflections, highlighting that mathematical learning is a social activity.
- **Guidance principle:** While students take an active role, the teacher plays a fundamental role, guiding the activity from a brief verbal introduction to the use of specialized software. The teacher's intervention ensures structured and coherent learning.
- **Reality principle:** The experiment focuses on everyday phenomena, such as the free fall of a ball, situations that students can imagine or have experienced. By connecting mathematics to real contexts, students can see the relevance and applicability of their learning.

Procedure for Data Collection

In this research, the challenge lies in recognizing the process of mathematization. For this purpose, the collection of various means has been proposed to deepen the explanations and reasoning of the students. These media include the students' activity sheets, videos of the collective sessions, audio recordings of the conversations within the work teams, photographs of the blackboard that capture the mathematical approaches, and the computer files generated by the Tracker software. The video recordings of the group sessions were made in a specially equipped classroom. Three cameras were placed at strategic points to capture the interactions between students and their interactions with the material. These recordings were made during all experiment phases to avoid missing key moments in the mathematics activity. Then, these recordings were reviewed and transcribed, focusing on the most relevant moments, where students expressed their mathematical reasoning or posed questions.

Since video recording does not allow the recording of local interactions in each team, audio recorders were used to record dialogues, debates, and spontaneous reasoning among students when approaching and solving problems. These recordings were made using recording devices placed in the work area of each team in order to avoid external interference. Before starting the sessions, students were informed about the recording and assured of these audios' confidentiality and strictly academic use. Subsequently, these recordings were transcribed to facilitate their analysis. The transcripts were labeled and classified according to the date, the problem or activity being addressed, and the specific work team. Through qualitative analysis, patterns were identified in how students verbalized their ideas, discussed resolution strategies, and how their responses evolved through the collaborative dialogue.

At the beginning of each session, students were given an activity sheet to complete individually, but they were encouraged to discuss their responses in groups. These sheets were then compiled and analyzed to identify patterns, common errors, effective strategies, and areas of confusion. In addition, comparisons were made between the written responses and the audio-recorded discussions to understand better how students verbalized and reasoned their responses compared to what they wrote down on paper. These activity sheets were digitized and stored in a database, labeled and categorized according to various criteria, such as date, activity topic, and student work team, to facilitate further analysis and correlation with other media.

The blackboard photographs were fundamental in visualizing the mathematical approaches developed during the sessions. These images captured the mathematical ideas as the discussions and collective reasoning progressed. By taking photographs at the end of each significant stage of the discussion process, the progression of mathematical thinking and proposed solutions could be visually documented. These images were cataloged and archived chronologically to facilitate their review and analysis with other media. On the other hand, the Tracker software made it possible to obtain detailed computer files of the students' mathematization process. These files were stored and labeled for later analysis and correlation with group discussions and worksheets.

Categories of Analysis

In order to study the process of progressive mathematization, four moments were established that were aligned with the tasks in each of the phases of the didactic activity. The first phase M1 (coded as progressive mathematization phase 1, situational) refers to the production of Cartesian graphs on paper from the verbal description of the experiment. M2: Analysis of the video of the experiment in the Tracker software and construction of the table of values (referential phase). M3: Confrontation and analysis of the initial graph with the one obtained with the Tracker software (general phase and internal validation 1). M4: Construction of the algebraic expression of the function from the table of values and comparison between the graph obtained with the function, the one obtained with the Tracker software, and the initial graph (formal phase and internal validation 2). At the end of each phase, the teacher organized a plenary session to analyze, reflect on and comment on the answers obtained.

On the other hand, four categories of analysis formulated from the theoretical approaches were established. For each category, some descriptors were proposed, understood as attributes or characteristics of a qualitative nature that allow the treatment and recovery of the data, which were established based on the theoretical discussion of the research.

The first category is called "function knowledge" formulated by Selter and Walter (2020), who argue that learning is a process of building knowledge and skills for a structured mathematical entity. The descriptors for this category are the development of a specific notation for the function (Sakja, 2003). The translation of information between different representation registers and fitting an algebraic expression to a table of values (Best & Bikner-Ahsbahs, 2017). The link between graph and function (Eisenberg, 2002). The relationship between variables (Falcade et al., 2007). Finally, qualitative interpretations of variables (Duval, 2006).

The second category, "study of problem situations," is formulated from the approaches of Kaur et al. (2020), who point out that mathematics studies start from problem situations in realizable contexts. The importance of physical experience in the study of problems as a means of reflecting on situations and translating the movement of graphs into physical movement (Arzarello & Robutti, 2004). Finally, the observation of experiments and formulation of conjectures (Sun & He, 2020).

The third category, "regularities and connections," refers to the "aha" moments students experience in developing the activity. Based on Selter and Walter (2020), formal thinking develops by discovering regularities, connections, and new structures. Descriptors for this category are the discoveries students make (Scherer, 2020) and the reinvention of mathematics (Wittmann, 2020).

The fourth and final category is called "progressive abstraction", which is taken from Selter and Walter (2020), who point out that learning is a process from the concrete to the abstract. Descriptors for this category use intuition to develop their strategies and use their own words (Van Den Heuvel-Panhuizen, 2020) as Selter and Walter (2020) pointed out, continuous reflection on their thinking processes and finally, the development and use of their strategies (Van Den Heuvel-Panhuizen, 2020).

The relationship between phases and the progressive mathematization process, is as follows: Phases M1 to M4 describe a sequence that goes from the situational, starting with a verbal description of the experiment and elaborating a Cartesian graph, to the formal, ending with constructing an algebraic expression of the function. This describes the process of progressive mathematization as it shows how students start with something concrete and familiar and gradually move towards an abstract and formalized understanding of that phenomenon.

The categories established in the research provide a structured framework for analyzing the process of progressive mathematization in students. The "knowledge of function" category focuses on how students conceptualize and mathematically represent functions, a fundamental basis of mathematical activity in the experimental situation since functions are a primary way of expressing mathematical relationships and modeling real situations. Specifically, aspects such as the development of notation, translation between different registers of representation, and deep understanding of relationships between variables reflect the student's ability to connect abstract ideas with concrete representations.

On the other hand, the "study of problem situations" highlights the importance of applying mathematics to tangible and realistic scenarios. Physical experience and direct observation of experiments strengthen the student's ability to translate real-world situations into a mathematical context, an essential aspect of mathematization.

The "regularities and connections" category raises the ability to discover and understand patterns, allowing students to connect previous experiences with new mathematical concepts. These moments of discovery, often referred to as "eureka" moments, are crucial to the mathematization process, as they indicate deep understanding and an ability to identify intrinsic mathematical relationships.

Finally, "progressive abstraction" describes the student's ideas development as they move from concrete mathematical concepts to more abstract understanding. This transition is essential in the mathematization process, as it indicates the development of higher critical thinking and reflection skills. By using their intuition and articulating strategies in their own words, students demonstrate an internalized understanding of mathematics and a metacognitive ability to evaluate and improve their learning process. This category, therefore, highlights the evolution of mathematical knowledge and how students process and refine that knowledge over time.

Participants

The didactic activity was carried out in a two-hour session with 21 third grade high school students in a rural school in Mexico. These students have a background in problem-solving and the study of situations involving linear variation models in their two previous courses. In addition, the students have worked with another experiment under the same approach and methodology reported in this article in which the variation of the height of the liquid in containers filled with water per unit of time is analyzed, leading to a linear model.

The teacher leading the session divided the students into seven working teams (E1, E2, ..., E7) with three members per team (each student coded A1, A2, and A3). The students organized themselves within the teams and distributed the tasks for setting up and obtaining data from the experiment, having their materials to experiment.

The teacher (T), who is part of the team of authors of this research, was careful not to indicate specific procedures or influence students' decisions. His primary role was to organize and coordinate the group sessions to encourage exchanging ideas and guide the discussions. In the first phase, the teacher verbally introduced the experimental approach, limiting himself to describing the variation conditions. In the next phase, he coordinated the experiment's setup and organized the students to be in charge.

During the video analysis, the teacher guided the discussion to identify and highlight the conditions of variability and the duration of the experiment, among other relevant aspects. In the third phase, he coordinated the discussion in which links were established between the shape of the graph and the experiment, proposed the "reading of the graph" to identify and describe the variation, and suggested the analysis of the data table to identify the position in the video.

In the fourth and fifth phases, the teacher coordinated the discussion for constructing the experiment's algebraic model and organized the validation of the results using GeoGebra.

RESULTS

Table 1 presents a concentration of categories with their observable descriptors during progressive mathematization. This "category-descriptor" analysis procedure allows a detailed process analysis and guides the data exploration. **Table 1** was constructed from the data analysis obtained in each phase, where the authors worked independently to generate a concentrate and were subsequently constructed collectively. It should be noted that, as shown in **Table 1**, not all descriptors necessarily appear in the progressive mathematization moments.

To illustrate how the descriptors were identified throughout the different phases, let us examine the case of "record translation." This descriptor refers to the ability to move from one representation of a mathematical concept to another. For example, in functions, this translation occurs between a graphical representation, its algebraic representation, and its representation in a table of values. We will analyze how this descriptor manifested in phases M1 to M4 of the didactic design.

Phase M1: In the initial introduction, the teacher verbally presented the situation of the free fall of a ball. The students were then asked to translate this verbal representation into a Cartesian graph. The translation of registers here was manifested when the students, based on their prior knowledge and conceptions, anticipated and drew the shape that the graph of the fall would have.

Table 1. Categories & descriptors observable in progressive mathematics moments

Categories	Descriptors	Observables			
		M1	M2	M3	M4
1. Knowledge of the function	1a. Specific notation				•
	1b. Translation of records	•	•	•	•
	1c. Table of values/adjustment to expression		•		•
	1d. link between graph and function	•		•	•
	1e. Idea of relationship between variables	•	•	•	•
	1f. Qualitative interpretation	•	•	•	•
2. Study of problem situations	2a. Physical experience		•	•	
	2b. Observation of experiments		•	•	
3. Regularities and connections	3a. Student discoveries			•	•
	3b. Reinventing mathematics			•	•
4. Progressive abstraction	4a. Intuition	•			
	4b. Continuous reflection			•	•
	4c. Strategy development			•	•

Phase M2: With the experiment filmed, students had to translate the visual observations from the video into a tabular representation. From the video measurements, they constructed a data table. This translation involved moving from a direct visual representation to a tabular one.

Phase M3: As the experiment was repeated and the Tracker software was introduced, the students were tasked with translating the raw experimental data into a graphical representation. The software generated graphs based on the position-time coordinates of the falling object. Here, log translation was particularly crucial, as students had to compare their initial graphs and predictions with the information generated by the software.

Phase M4: In this last phase, students analyzed the resulting graph from the software and compared it with their initial graph. In addition, they engaged in the process of translating that graphical representation into an algebraic representation. The ability to move between these representations and understand how they relate to each other was essential to each instructional design phase.

Categories & Descriptors in Each Mathematics Phase

During the M1 phase of the didactic activity, students were challenged to anticipate the graph from the physical experiment to reflect on the conditions of the experiment, and to put into play their intuitive ideas about the relationship between variables, the reference system, and the shape of the graph. This task involves translating information from the context of the problem into mathematical expressions. It was found that three of the seven teams drew a vertical line on a plane to describe the experiment. However, team E1 proposed a graph describing the trajectory of a falling ball, leaving a trace on the plane. This is consistent with the results of other studies suggesting that students tend to represent the direction of motion rather than the passage of time (Clement, 1989). Although the participants have been studying functions for the last three years, their ability to express relationships of quantities through graphs still needs to be improved, highlighting the complexity of identifying or choosing the variables of the situation and expressing them through a graph indicating the relationship between variables.

Regarding the qualitative interpretation of the situation, the E3 team's argument about the curvature of the graph, referred to as the "linearity of the curve" stands out. This argument assumes that the ball falls at a constant speed, which leads to a straight line with a negative slope. Although this assumption is not valid, it made it possible to argue for the increase in velocity. Once this is admitted, very short intervals of time are considered in which a straight line is defined for each interval, allowing a curvature to be observed, as illustrated in **Figure 2**. This intuitive explanation of the motion was discussed in the plenary, where the E3 team presented their ideas.

> E3-A1: Suppose [the ball] is at five, and if in the first second it advances 10 centimeters, in the second 10 centimeters, you get a straight line downwards.
>
> T: So, you agree that you get a line with a negative slope?
>
> E3-A2: No, no, it is just that ... if it is OK for a short time, then there is the change to another speed and another straight down [steeper slope].
>
> E3-A1: The idea is that the graph can be seen as the union of these lines. Of course, if you compare it with a curve, you have to make the lines smaller.

The idea of E3 is to draw a succession of straight lines in short time intervals with a clear tendency to a curve, as de L'Hôpital (1768) argued in the 18th century when he explained the infinitesimal nature of curves: we ask that a curved line can be considered as the set of an infinity of straight lines (p. 3). This argument arose as an extension of what was discussed in situation 1 (not reported in this article), where linear behavior was analyzed in a container-filling experiment. This experience shows a process of mathematical reinvention stimulated by the situation and the handling of different registers of representation.

These results offer a deep perspective on how students conceptualize and visualize complex physical phenomena. Although based on an incorrect assumption, this interpretation highlights the students' ability to reason and adapt their previous

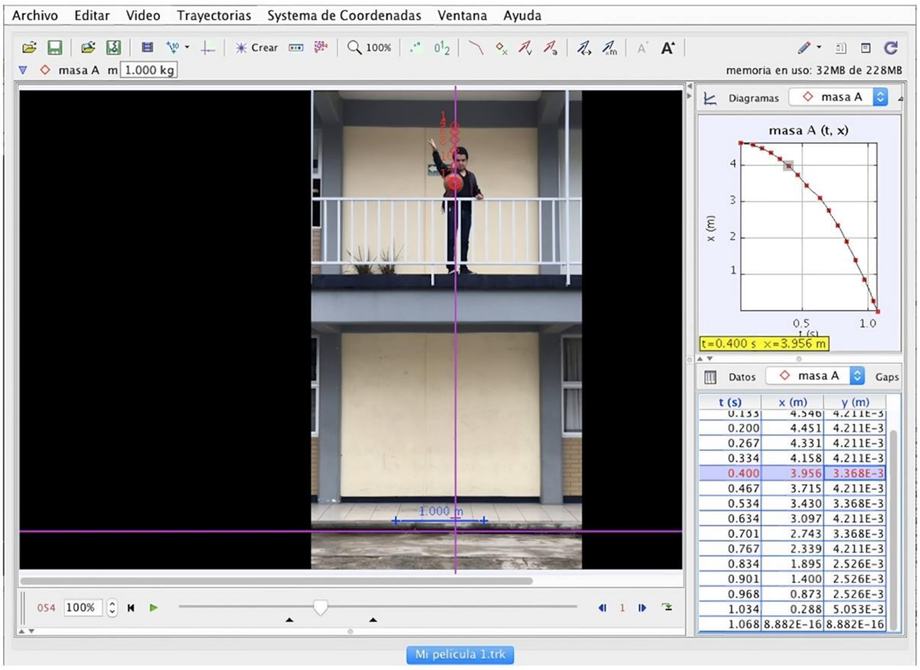

Figure 2. Free-fall experiment of a ball, table of values, & graphs generated with software (Source: Field study, reprinted with permission of the participants)

knowledge to new situations. Visualizing movement as a succession of straight lines in short intervals of time reflects an intuitive understanding of motion. It resonates with historical mathematical theories, such as the one proposed by de L'Hôpital (1768) in the 18th century. These findings underline the importance of allowing students to explore and discuss their interpretations, even if they initially seem inaccurate. These discussions can serve as starting points for building a deeper and more formalized understanding.

In phase M2, students filmed the experiment. Then they analyzed the video to observe the conditions of the situation, identify the variables and their relationship, and the duration of the experiment, among others. The physical experience allowed the students to control the data collection and intuitively recognize the variables and the interval, where the experiment occurs. In addition, by observing the experiment, a specific value can be related to the ball's position. They constructed tables with values from these reflections to establish the relationship between distance and time.

In order to generate the table of values, the students used a standard scale that was included in the film. In this way, they could determine the ball's position relative to the ground (in the film) for each unit of time *t*. The procedures were diverse, some measuring the standard scale with a ruler directly on the screen, others using a pixel scale, and others with strips of paper superimposed on the screen. This activity represented an exercise in translating information in different registers, from the physical situation (recorded on video) to a table of values, where the relationship between the variables distance and time was also observed. All teams managed to establish their tables of values.

These results reveal the students' ability to connect physical experiences with abstract mathematical representations. By filming and analyzing the experiment, the students acquired practical data collection skills and an intuitive understanding of the variables at play and their interrelation. When translating information from a physical situation to a table of values, the students perform "horizontal mathematization," connecting the real world with the symbolic. This skill reinforces their understanding of the concept of function and equips them with transferable skills to address complex mathematical problems in varied contexts in the future.

In terms of the relationship between variables, it was observed that the idea of "time" emerged spontaneously in this activity, in three different scenes, when the teams started to manually set the points to indicate the height as the experiment occurred. Through some questions that the teacher posed to guide the qualitative analysis, for example, at time zero, how high is the ball? At second 1, how high is the ball? Moreover, by working with the video itself, the film is associated with a "timeline" in the digital player, indicating a time scale t.

In the plenary session, the teacher encouraged a reflection on the information provided by the tables of values about the experiment. Team 4 shared a reflection that alludes to the interval, where the experiment occurs.

> E4-A2: ... The experiment is only presented from five meters to the ground, which is zero meters, and from zero time to approximately three seconds.
>
> T: What happens outside these values?
>
> E4-A1: No information because the experiment only occurs at these values.

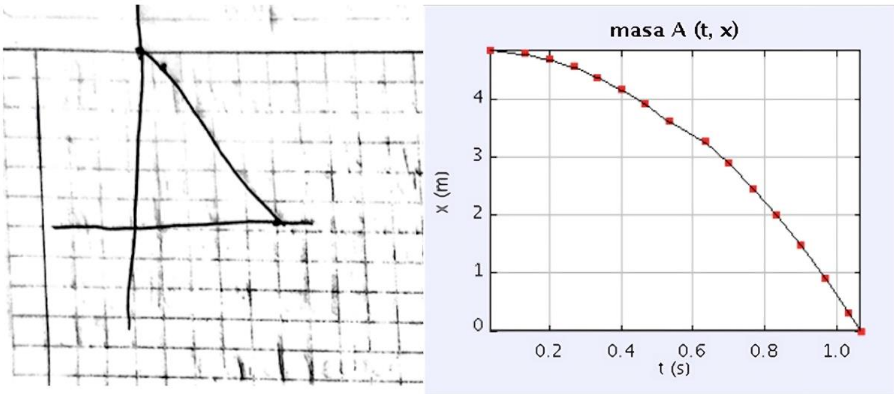

Figure 3. Proposed graph & graph obtained in phase 3 (Source: Field study, reprinted with permission of the participants)

These results reflect the student's ability to identify and relate variables in a practical context. This experience allowed them to visualize and intuitively establish the relationship between time and height, identifying how one variable affects the other in a practical and realistic context.

In phase M3, the task consisted of confronting the initial graph plotted on paper with the one obtained by the software, as shown in **Figure 2**, leading to a comparison and reflection.

The teacher organized a plenary session, where the students presented their initial graphs and compared them with those obtained with the software to motivate reflection and analyze differences and similarities. This validation of the results was a decisive stage in the situation, as the feedback allowed the students to consolidate their intuitive interpretations of the experiment, or for those who presented erroneous graphs, it allowed them to rethink their initial ideas.

Regarding the translation of registers, the plenary session saw the students reflect on "physical height", understood as a reference point for analyzing the ball's movement. Establishing the point of origin made it possible to follow the experiment through the video, the table of values, and the Cartesian graph. The following dialogue shows the identification of the origin in terms of the experiment.

E6-A3: The ball falls from a certain height, so we are only interested in what happens from height to the ground.

T: What would you call that point?

E6-A2: Well, it is the starting point of the experiment, or in the video, the table, and the graph, they are all at zero, zero time, zero distance, well, it is five meters high, but it still does not advance any centimeters.

T: Is that point important?

E6-A1: This is, where the experiment starts, where we start taking data.

One feature offered by the software is the relationship of the film (on which the analysis is performed) with the data table and the graph, which allowed the students to look at the experiment from three different representations, all sharing the exact information of the experiment. Explaining the experiment from three different representations helped to deepen the understanding of the situation and strengthen the meaning of the mathematical representations.

The results of this phase demonstrate the importance of confronting intuitive interpretations with advanced technological tools in the learning process. This confrontation allowed the students to validate, consolidate, or reassess their initial interpretations, strengthening their understanding of the studied phenomenon. Students gain a deeper and diversified understanding of functions by relating and comparing different representations (video, table of values, and Cartesian graph).

About the link between graph and function, an interesting discussion was observed from E7. They pointed out that the initially proposed linear and decreasing graph was different from the correct one. However, they commented that both are decreasing and express similar behaviors. The following dialogue shows the comparison between the two graphs.

E7-A3: Yes, they are different, but they both go down [see **Figure 3**]. They start the same and end the same [refers to coordinates].

E5-A1: But the first one is ... straight and the other one is curved.

E7-A3: Yes, but I mean ... they have a similarity.

E7-A2: From 0.6 to one [in the graph generated by the software], it looks like the one we drew. They are the same situation; well, before 0.6, you can see that it is a different movement.

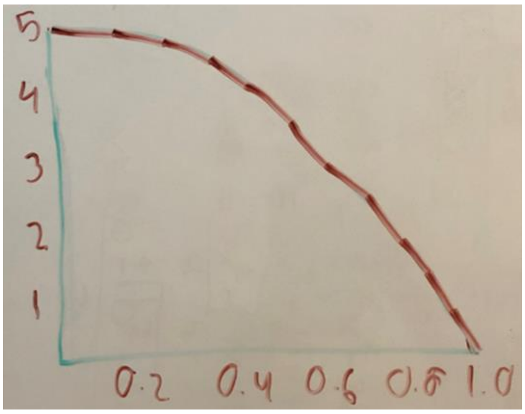

Figure 4. Curve made up of straight segments (Source: Field study, reprinted with permission of the participants)

Several teams expressed their bewilderment in the plenary at the graph they obtained with the software. Three teams were convinced that the graph would be a straight line with a negative slope, but when they observed that the graph generated by the software was a curve, they repeated the experiment twice, thinking that they had done something wrong. Extensive discussion was generated to explain the curved nature of the graph qualitatively. Team 3 argued that the situation did not lead to a straight line, which prompted an intense dialogue to explain the graph's curved nature.

E3-A1: It is assumed that when the ball falls, it is supposed to increase in speed.

T: How did you conclude that? Can you discuss it with colleagues?

E3-A2: At the beginning of the experiment, the ball does not move, and therefore it has no speed. In the first second, it has already moved forward and has speed, and then by the second two, the ball is already going faster. You can see it when the ball bounces; when the bounce is higher, it is because it is coming with much speed.

This explanation was illustrated by bouncing a ball on the spot, i.e., again relying on physical experience to explain the graph, first by dropping the ball from two different heights and looking at the height of the bounce. Then, adding force to make the ball bounce on the ground with more incredible speed, thus achieving a greater height. This new physics experiment extends the initial one in which the movement phenomenon with different velocities is analyzed. The students observed the experiment again, and this allowed them to strengthen the idea of different speeds in the fall, which represents a discovery on the part of the students and an opportunity to reinvent mathematics. The discussion progressed with the intervention of E3, who presented the idea of successive straight lines (**Figure 4**).

E3-A1: It is supposed to be the same as [happened] in the other problem [referring to the previous activity of filling containers] because you can see that in the first second, it goes at one speed, which is a straight line, in the other second it [goes] at more speed and gives another straight line and if you put all the straight lines together it looks almost like a curve.

The results reflect how students, when interacting with real or "realizable" situations, develop a deep understanding of the relationships between variables, such as time and position. The discussion around the curved nature of the graph and the idea of successive straight lines evidences a process of "progressive mathematization," where students move from a situational understanding based on tangible experiences to more abstract and formal levels of mathematical reasoning. For the students, this represents an opportunity to reinvent concepts based on their interpretations and discussions mathematically. In addition, it is identified that the relationship between variables becomes an essential tool for learning about functions.

The last phase M4 was dedicated to the construction of an algebraic expression from the information available in the Tracker software (the video with the points, the data table, and the graph, as shown in the graph in **Figure 1**) and the dialogues in plenary sessions, where the teacher motivated a reflection on the work done.

A critical background was that the students had addressed in their classes the topics of problem-solving through the formulation and algebraic solution of linear and quadratic equations and the study of situations of linear variation from their tabular, graphical, and algebraic representations (Secretaría de Educación Pública [Secretary of Public Education] [SEP], 2017). For the stage of validating the algebraic expression, the teacher instructed the students to place in GeoGebra all the points obtained in the Tracker software and plot the proposed function on this point cloud to ensure that the graph represented the points. The idea of a representative curve was the subject of much discussion because, for the first time, the students were in a scenario, where they had to approximate the empirical data as closely as possible, something that does not usually happen in mathematics class.

This task of fitting an expression was very complex for the students. Only four teams managed to pose an algebraic expression; of these, only three did it correctly. Two teams used the "trial and error" strategy on GeoGebra. They chose a quadratic function and modified the coefficients to fit the curve to the data cloud. However, the procedure followed by E3 stands out. He used an algebraic method studied in the previous school grade (SEP, 1999) to determine the algebraic expression from a numerical sequence, or in this case, coordinates. In this episode, we observe the appearance of a specific notation to refer to the function. First, they chose three pairs of points (x_1, y_1), (x_2, y_2), and (x_3, y_3) from the data table, then they used the general equation of second-degree $y=ax^2+bx+c$ to construct a 3×3 system of equations, thus forming the following 3×3 system of equations:

$$y_1 = ax_1^2 + bx_1 + c$$
$$y_2 = ax_2^2 + bx_2 + c$$
$$y_3 = ax_3^2 + bx_3 + c$$

It is important to note that although the system has three equations and three unknowns (*a*, *b*, and *c*), it is not "linear" in the traditional sense because of the quadratic terms (such as *a*×2). However, concerning the variables *a*, *b*, and *c*, the system behaves linearly. If we consider the "*x*" as constants (based on the data collected), the system is linear in the unknowns *a*, *b*, and *c*.

The above system of equations can be represented as a matrix multiplication, as follows:

$$A = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}$$
$$X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

where $AX = Y$.

However, they were soon overwhelmed by the complicated calculations when trying to solve the 3×3 system of equations. The teacher intervened to suggest using the Alpha Wolfram platform to quickly determine the coefficients of the function instead of getting bogged down in the lengthy calculations. The group exchange focused on analyzing the procedure followed by team 3 and its validation through GeoGebra. After verifying that the parabola obtained passes specifically through the three chosen coordinates, there was a broad discussion on the possibility of obtaining different functions from the coordinates. With this method, three teams succeeded in obtaining their function. This episode shows the rediscovery of an algebraic method to determine an algebraic expression, which represents the use of various strategies to tackle the problem and a process of reinvention of the studied mathematics.

The idea of "function as a representation of something" also appears in this scene. E6 shared a reflection, where the relationship between graph and function is observed, emphasizing the translation of registers between physical experience, numerical tables, graphs, and algebraic expressions.

> E6-A1: It all refers to the same thing. In the experiment, in the graphs, in the table, and in the function [referring to the formula], it is all the same things. If you want to see something, you can see it in any of them.
>
> T: Which is the most important?
>
> E6-A1: Is when we saw functions, we only studied how to graph and find points, but now I see that everything is related, but what I'm left with is that the function is something that represents something like a situation.

To conclude this phase, the teacher asked the students to derive the graph from the free fall equation of kinematics $y = h - \frac{1}{2}gt^2$, where *h* is the height, *g* is the acceleration constant of gravity, and *t* is the time. The purpose was to compare the graph with those obtained previously and to relate the experiment, which emphasized the identification and relationship between variables, with what they had studied in their secondary school physics classes in which the subject of free fall is addressed, but usually from an algebraic approach. During the plenary session, E7 contributed a reflection, transcribed in the following dialogue, on the similarity of the graphs obtained, where the qualitative interpretation of the experiment is articulated with the algebraic expressions.

> E7-A2: When we looked at our graph [decreasing straight line] in the ball drop, we were sure it was a straight line, but no, it was actually a curve. However, if you see, they look very similar because they go down. Here in the free fall graph [obtained from the free fall equation], at first, I saw it differently, but actually, they have many similarities, starting because they are quadratic.

The results obtained in phase M4 reflect the deep connection between practical experience and mathematical concepts. This activity allowed students to visualize, experiment with, and recognize the idea of a relationship between variables in a concrete context. This notion of a relationship between variables is fundamental in defining a function from Dirichlet's perspective, as it justifies the variation of the function based on changes in one of the variables. Within the framework of RME, it is promoted that learning emerges from students' mathematical activity rather than memorization of pre-fabricated axioms. Therefore, by relating variables in a real context, students engage in "progressive mathematization." These findings highlight the importance of allowing students to explore and discuss their interpretations, connecting mathematical definitions with practical situations. The impact on learning is significant, as when faced with the task of adjusting an expression and rediscovering an algebraic method, students demonstrate mathematical inventiveness, reflecting the use of various strategies to address the problem.

CONCLUSIONS

This study explored the progressive mathematization of the concept of a function in high school students, using an experimental free-fall activity to foster the development of fundamental ideas of the concept of a function, such as relationship and variation. The research question focused on discerning how mathematical modeling, from a RME perspective, contributes to the understanding and development the concept of a function.

Prior literature has consistently identified students' difficulties with the concept of a function. These difficulties stem from the epistemological nature of the concept, which intertwines various mathematical concepts and traditional pedagogical approaches that often limit its teaching. As Breidenbach et al. (1992) and others noted, understanding a function goes beyond mere algebraic manipulation. It is a multifaceted concept that requires a deep understanding of relationships, graphical representations, and real-world applications.

Despite students' familiarity with a function after years of study, challenges persist when attempting to express quantitative relationships through graphs and linking these representations with physical phenomena and algebraic representations. In this context, the proposal to address the concept of a function through a didactic free-fall ball activity proved to be an opportunity to establish variable relationships. Furthermore, this activity promoted students' transition from informal strategies to formal procedures and more abstract mathematical representations. Particularly noteworthy is the emergence of the "linearity of the curve" category, reflecting a versatile interpretation of graphs and strengthening the relationship between the observed phenomenon and its graphical representation. This direct connection between experience and representation is essential for RME. It aligns with the recommendations of previous researchers such as Arzarello and Robutti (2004) and Ortega et al. (2019), who have emphasized the importance of physical and experimental experiences in mathematics education.

The analytical framework proposed in this article, consisting of observables (categories) and descriptors, allowed for a detailed exploration of the progressive mathematization process. Through this framework, the phases and characteristics of the mathematization process were identified and described, from informal strategies associated with the context of the problem to formal procedures. The main contribution of this article to RME has been to provide a detailed structure and analytical framework for understanding and analyzing the progressive mathematization process in the context of function teaching. It has highlighted the importance of rich and realistic situations, student interaction, and reflection on the learning process.

Although fewer than half of the teams were able to establish an algebraic expression, collective work facilitated reflection on the solution processes that emerged within the teams, providing an opportunity to interconnect different conceptual elements from prior knowledge, mathematical ideas that emerged in the situations, and the articulation of mathematical representations provided by the software.

Another observed characteristic in this mathematization process is the absence of unique answers and procedures; students presented both informal and sophisticated solutions. However, this diversity of responses, drawn from each student's reality and thinking, allowed for discussions about the nature of each procedure, showcasing arguments closely associated with experiments or advanced explanations of the local behavior of curves.

In conclusion, this study highlights the imperative need to integrate experimental activities and technological tools into teaching complex mathematical concepts. This integration enriches the learning experience and provides students with a broad contextual framework for applying mathematical knowledge in real situations. Progressive mathematization, evident in this study, is a dynamic process that demands a reflective pedagogical approach, suitable tools, and an open disposition on the part of students.

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