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Mathematics education: What was it, what is it, and what will it be?

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ABSTRACT

The evolution of mathematics coincided with advancements in its teaching. The 19th and 20th centuries marked a pedagogical revolution in mathematics education. This paper argues that Bruner's (1966) model, Gagné's (1985) taxonomy, innovative teaching methods emphasizing research and problem-solving, and the inclusion of data analysis topics have shaped modern mathematics education. Additionally, the paper explores transformative trends, emphasizing mathematics literacy and the integration of virtual reality (VR) and artificial intelligence (AI) in education. This evolution emphasizes practical, contextually relevant approaches. VR enhances engagement and comprehension of abstract concepts, while AI offers personalized learning experiences, fostering deeper understanding and skill development.

Keywords: digitalization, history, mathematics education, mathematics literacy, teaching

INTRODUCTION

Mathematics is indispensable for development of every nation, both as a fundamental element for science and technology (Adenegan, 2011), and for the development of the human being (Felda & Cotič, 2012). Mathematics involves logical thinking, rationalization, knowledge, and application (Adenegan, 2011; Bone et al., 2021). The teaching of mathematics, which has started since the birth of mathematics, makes use of numerous tools and materials to facilitate learning (Oyekan, 2000). Mathematics is the study of numbers, sets, and other abstract entities, along with the relations and operations between them (Adenegan 2011). It is a model of thought used in sciences to draw conclusions. Initially, school curricula focused primarily on arithmetic so that people could "do math". Then, in the early 1950s, mathematics began to be seen as divided into three branches (Adenegan, 2011):

(1) arithmetic,

(2) algebra, and

(3) geometry.

Much has been written about the history of mathematics, as it is one of the oldest (Man-Keung, 2000) and most useful sciences for humanity (Chesky & Wolfmeyer, 2020). We know that mathematics began to develop in the Neolithic period and then rapidly progressed in ancient civilizations in China, India, Babylon, Egypt, Ancient Greece, and the Roman Empire, continuing to evolve to this day (Boyer, 2011). Alongside mathematics and the development of society and various sciences throughout history, mathematics education has also evolved (Kilpatrick, 2020), which is the topic of the present contribution. The aim of this paper is to succinctly outline some key ideas that have shaped current trends in mathematics education, encompassing both pedagogical approaches and content-related considerations. Specifically, we assert that there has been a shift in interest within mathematics education from a purely practical perspective to one that is more reality-oriented, emphasizing problem-solving. Furthermore, the incorporation of new topics such as statistics and probability has significantly impacted national curricula worldwide. Additionally, Bruner's (1966) learning model and Gagné's (1985) taxonomy have made notable contributions to the advancement of mathematics education. While digital technology is now prevalent in mathematics education, we contend that virtual reality (VR) and artificial intelligence (AI) also hold promise for future integration into teaching practices.

HISTORY OF TEACHING MATHEMATICS

From Antiquity to Enlightenment

For mathematics, number was at the beginning (Boyer, 2011). Even today, many mathematicians believe that number is not only the historical beginning of mathematics, but–more broadly and fundamentally–its deepest foundation. Thus, its first

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beginnings are seen in counting, in the emergence of the concept of "abstract natural number", in the formation of names and symbols for numbers, and later in the first steps towards elementary computation with such introduced numbers (Boyer, 2011).

Skills in calculation were taught even in the earliest times before our era. This is confirmed by ancient documents found in the temple of the god Baal in Nippur, which bear symbols of numbers and arithmetic operations (Friberg, 2000). These documents date back to several millennia before our era. Instructions for arithmetic operations are also found on ancient Egyptian monuments and papyri (Peet, 1931b). The Egyptians used a counting board called an abacus, which had movable stones, for calculations. The Romans also used a similar abacus and introduced it for practical use throughout their empire (Sugden, 1981). The Egyptians also had a highly developed geometry due to practical needs in measuring fields along the Nile and in construction (Peet, 1931a). Based on what we know today about ancient Egyptian mathematics, it seems they considered it an empirical science. Mathematics only became a deductive science in ancient Greece (Logan & Pruska-Oldenhof, 2022).

In the Middle Ages, all arithmetic instruction took place on abacuses (Evans, 1977). Without their assistance, solving calculations with "large" numbers was very difficult, as positional numeral systems were not yet known (Pisano & Bussotti, 2015). The positional numeral system, or the Indian-Arabic numeral notation in decimal composition, had a significant impact on the development of mathematics and culture in general. However, despite its advancement, mathematics was not introduced into schools as a compulsory subject for a long time. In the Middle Ages, "arithmetic masters" emerged in cities who taught computational skills (Ulivi, 2016). The most famous medieval arithmetic master was the Frenchman A. Riese (1492-1559). Arithmetic in the Middle Ages was taught solely for practical needs in crafts and trade (Browder, 1976).

Arithmetic instruction was introduced into elementary schools as a compulsory subject only towards the end of the 17th century (cf. Ellerton & Clements, 2022). Until the second half of the 18th century, the teaching of arithmetic and geometry had only one task: to develop mechanical skills in solving practical problems (Malaty, 1998). Therefore, instruction took place without explanation and logical reasoning. Only with the Enlightenment did they begin to emphasize formal goals and tasks of arithmetic instruction, as they wanted arithmetic to develop individual mental functions (Henry, 1995). Therefore, they aimed for computation with understanding, which introduced the principle of clarity into arithmetic instruction and, accordingly, various didactic tools. Unfortunately, despite all these efforts, school practice remained primarily rote learning and memorization of arithmetic rules in most cases.

From Enlightenment to 20th Century

The first breakthrough of the direction that demanded arithmetic instruction to fulfill educational tasks was established in pedagogical practice only with the educator and psychologist J. H. Pestalozzi (1746-1827) (Mesquida et al., 2017). For him, arithmetic instruction was truly education and the development of young people, not just rote learning. Although Pestalozzi exaggerated with arithmetic exercises in special tables and delved into excessive arithmetic formalism (Hartung, 1962), we must recognize the merits he has for the development of arithmetic instruction. His works inspired numerous educators (e.g., Tillich, Harnisch, Diesterweg, and Grube) of the 19th century to begin studying and developing arithmetic instruction, thus continually improving it (cf. Bullynck, 2008). The principles of arithmetic instruction of 19th century educators were largely considered by the pedagogue E. Hientschel at the beginning of the 20th century. His arithmetic calculations served as a model for arithmetic instruction for many years. He introduced a concentric arrangement of teaching content and thus made a significant contribution to the meaningful construction of the curriculum, especially for arithmetic.

From Past to Present

A significant milestone in the teaching of mathematics at the beginning of the 20th century was the conference in Merano in 1905 (Hamley, 1934). The gathered scientists attempted to adapt arithmetic to children's psychophysical abilities, considering the results of youth psychology and experimental pedagogy. Above all, they wanted arithmetic and geometry instruction to develop functional thinking, concrete perception of spatial relationships, and to discover mathematical relationships in nature, society, and life (Jahnke et al., 2022). The scientists who significantly influenced the development of mathematics in the first half of the 20th century were V. Prihoda, J. Kühnel, and J. Wittmann (Beyer & Walter, 2014). Despite their efforts to improve mathematics instruction, we can conclude that mathematical instruction until the middle of 20th century was limited to the necessity of providing students with indispensable tools for practical activities; that is, acquiring the necessary techniques for the four basic arithmetic operations. Thus, students learned only "practical" arithmetic. Geometry was less important; its instruction was narrowed down to planning shapes, making models of solids, calculating the perimeter and area of shapes, as well as the surface area and volume of solids.

Present

The year 1950 marks the beginning of significant changes in the teaching of mathematics worldwide. In that year, scientists, including psychologist J. Piaget, mathematician G. Choquet, and educator C. Gattegno, formed the CIEAEM commission: *Comission Internationale pour l'Etude et l'Amelioration de l'Enseignement des Mathematiques* [*International Commission for the Study and Improvement of Mathematics Teaching*] (De Bock, 2023). They extensively discussed modern mathematics and developmental psychology, giving rise to the idea of a radical reform of mathematics instruction at all levels of schooling. After 1960, renewal processes began in most countries (the renewal had an international character), involving mathematicians, psychologists, educators, and teachers (Furinghetti et al., 2012). Various tendencies emerged during the reform of mathematics education, often conflicting and divergent. This renewal took place in three phases (Furinghetti & Giacardi, 2023):

1. changes in content,
2. changes in teaching and learning approaches, and

3. realization that not only content and approaches matter, but above all, the child for whom content and approaches are intended.

In the first phase, it was believed that it was sufficient to add only new content to mathematics instruction. At the turn of the 19th to the 20th century, a new branch of mathematics emerged (Dauben, 1979): Georg Cantor's set theory (1845-1918), which quickly permeated most areas of classical mathematics, not only with its symbolism and terminology but also with its often novel methods. Set theory spread worldwide, even into mathematics education at all levels of schooling. There was talk of "new mathematics" or "modern mathematics" or "première vogue", as mathematician Z. Krygowska called it (Semadeni, 2023). With the introduction of new content into schools, there was dangerously excessive enthusiasm and heavy formalism was introduced (De Bock, 2023). Sets with logic appeared as early as the first grade; the concept of number was introduced to students as the cardinality of equipollent sets, arithmetic operations were based on operations among sets, and geometry was introduced as the study of topological transformations (Oller-Marcén, 2022). At that time, it was "forgotten" that the part of set theory that could be successfully used in elementary schools was extremely elementary, and children at this stage of development could not grasp more advanced set theory concepts (Ferreirós, 2008).

With *international commission on mathematical instruction* congress in 1972, where R. Thom sparked controversy with his lecture *modern mathematics: does it exist?!* (Thom, 1973), the second phase of changes in mathematics instruction began. Research conducted at elementary schools worldwide at that time showed that children were clumsy at calculation and poor at solving mathematical problems related to real-life situations (Furinghetti & Giacardi, 2023; Furinghetti et al., 2008; Schoenfeld, 2007). Based on these studies, it was found that it was naive and pointless to expect that the introduction of modern content alone would suffice to improve mathematics instruction while neglecting teaching and learning approaches. Thus, in Western Europe and North America initially, and somewhat later worldwide, new teaching and learning approaches began to be introduced, theoretically supported by the works of psychologists such as Piaget, Bruner, Bloom, Galperin, Gagné, and others (Feinstein, 1972). In this phase, the principles of personal activity or active involvement of the child gained importance in mathematics instruction. It became necessary to introduce mathematical content with activities based on children's exploration. Although mathematics instruction improved in the second phase of changes by refining content and using forms and methods suitable for students, mathematics was still taught as a discipline closed in on itself.

Some of the most important innovations introduced at this time, mainly based on the theoretical principles of cognitive and constructivist teaching and learning theory, were:

(1) Bruner's (1966) model of acquiring mathematical concepts,

(2) Gagné's (1985) taxonomy of mathematical content,

(3) modern teaching and learning methods (research-based and problem-based instruction, experiential learning, active learning, etc.), and

(4) introduction of data processing into mathematics instruction (basic elements of statistics, probability, and combinatorics).

Bruner's model of creating mathematics concepts

The student at the elementary level, in the process of acquiring new concepts, »transitions« to the following levels, which we have adapted and further developed according to Bruner's (1966) model of mathematical concept development. These levels are represented in **Figure 1**.

We must not view the enactive, iconic, and symbolic levels as if the process of concept learning occurred first only enactively, then iconically, and finally symbolically. We must conceive these different representational levels very flexibly and incorporate them differently into teaching; these three levels can follow each other sequentially, for example, concrete enaction can be transferred into an image, and then the image can be described and written with symbols, and also in such a way that concrete enaction is immediately transferred to the symbolic level. Furthermore, we can "operate" concretely based on the image and vice versa, describing the process based on the image and writing it with symbols.

Gagné's taxonomy

We describe students' achievements based on the level of acquired knowledge using a taxonomic scale, as depicted in the table below (Gagné, 1985; see **Table 1**).

Basic skills encompass a range of competencies, primarily centered around grasping concepts and recalling information (Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001). Conceptual skills, on the other hand, involve a deeper comprehension of concepts and facts. The fundamental components of foundational and conceptual skills comprise: familiarity with specifics (such as multiplication techniques, isolated information, and factual data); acquaintance with specific facts (including definitions, formulas, axioms, theorems, relationships, and basic properties); understanding of terminology and fundamental symbols (such as parallelism, orthogonality, +, -, %, etc., as well as terms like rectangle, function, equation, kilogram); awareness of classifications and categories (recognizing different mathematical objects and their categorization); identification of concepts (e.g., recognizing triangles in geometric shapes, solids, etc.); visualization (e.g., understanding that two congruent right-angled triangles can form a right-angled triangle); comprehension of terminology and symbolism in context; and establishing connections (identifying similarities, differences, and integration among concepts).

Procedural skills encompass the understanding and proficient application of algorithms and processes (Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001). This domain can be further categorized into routine procedural knowledge and complex procedural knowledge. The foundational components of procedural knowledge include executing

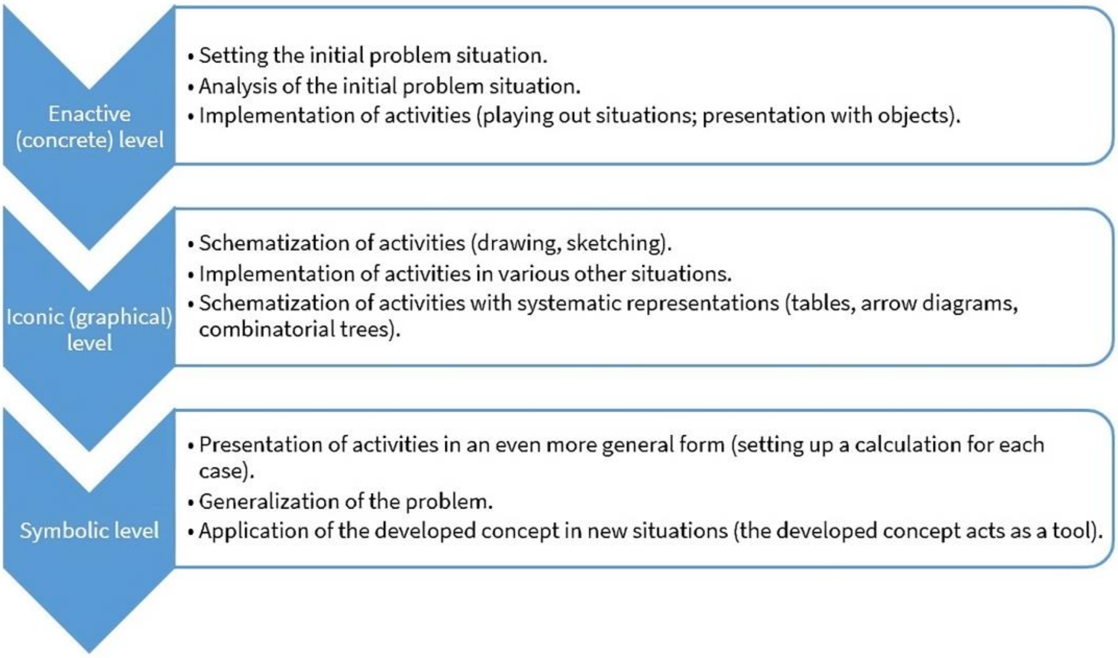

Figure 1. Model of creating mathematics concepts (Adapted from Bruner, 1966)

Table 1. Gagné's (1985) taxonomy of knowledge

Taxonomy level	Description
Basic & conceptual skills	Basic skills and knowledge; conceptual skills
Procedural skills	Routine procedural skills; complex procedural skills
Problem skills	Problem-solving strategies; applicative skills

standard procedures, employing rules and formats, solving straightforward tasks with minimal data, mastering algorithms and procedures effectively, applying rules and procedures without merely recalling them, selecting and executing procedures while justifying or verifying the selection, and applying complex procedures to handle composite tasks with multiple data sets.

Problem-solving skills are utilized for the application of knowledge in novel contexts, employing combinations of various rules and concepts to address new situations, and effectively applying both conceptual and procedural skills. The essential elements of problem-solving skills involve defining a problem (identifying and formulating the problem, posing relevant questions), verifying data (assessing if there is sufficient or excessive data, identifying any contradictions), devising solution strategies (utilizing communication, operational, thinking, and note-taking processes), applying skills or transferring knowledge (using mathematical concepts in different contexts), employing thinking skills (analysis, synthesis, induction, deduction, interpretation), and utilizing metacognitive skills (evaluating the correct application of mathematical concepts in specific contexts, justifying one's reasoning to demonstrate resolution of cognitive conflicts).

Knowledge influences each other: knowing procedures somewhat affects the understanding of concepts (Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001). Even in application, we never use just procedural or problem-solving knowledge, but both types of knowledge are intertwined. Therefore, it is impossible to assign greater importance to one type of knowledge over another, as they are so intertwined that they cannot be easily separated. The significance of various types of knowledge depends on external circumstances, the purpose of education, the subjective judgment of the teacher, etc. The interconnection between different types of skills is illustrated in **Figure 2**.

New learning methodologies

Research-based learning: Barron and Darling-Hammond (2010) describe research-based learning as project-based learning (where the term "project" represents a broader set of learning experiences), problem-based learning, and learning through design. Rocard et al. (2007) emphasized the need for a transition from deductive to inductive approaches in teaching in their recommendations for research-based learning. This approach, when properly guided by the teacher, allows for more observation, experimentation, and self-construction of knowledge by students.

Research-based learning is defined as a combination of theoretical knowledge and understanding with practical knowledge, skills, and abilities (Maaß & Artigue, 2013). The approach is based on several principles: direct experience is the path to understanding; students must understand the research question or problem, so they must define it, determine it, and master it themselves; research-based learning develops procedural knowledge (skills and abilities); research-based learning is not just manipulating materials, tools, etc., it is primarily a mental activity; the use of secondary sources complements direct experience; research is collaboration, it is group work.

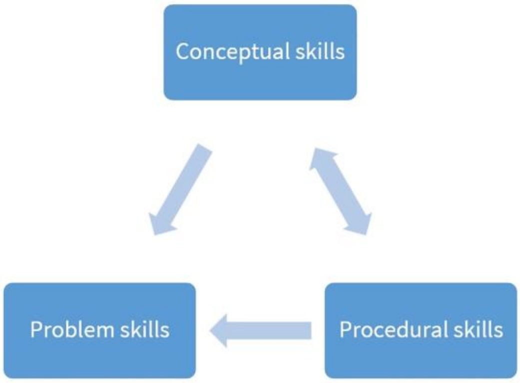

Figure 2. Relationship between different types of knowledge (Source: Authors' own elaboration)

In research-based learning, students: identify problems, critically evaluate, distinguish between alternatives, plan research, explore assumptions/hypotheses, seek information, construct models, discuss with peers, formulate coherent arguments, conclusions (Linn et al., 2013).

Within the framework of teaching mathematics through research, a problem is more than just a specific task, exercise, or activity. The problem is open in the sense that it requires active participation of students in experimenting, formulating hypotheses regarding possible solutions, communicating hypotheses and possible solution strategies, and perhaps also in asking additional questions that will need to be answered during the process of problem-solving.

A challenge for research-based learning (Antoliš et al., 2019): *If we uniformly increase all three lengths of a triangle, how much larger will the area of the enlarged triangle be compared to the area of the original triangle?*

Approaches to solving and researching the problem depend on prior knowledge (about triangles, measures of sides, angles and areas, similar triangles, and trigonometric ratios, etc.). Students can approach enlargement and experiment with it using addition and multiplication. They can create a large number of triangles, enlarge them, collect results, and formulate hypotheses about the increase in area. Furthermore, they can explore special cases (e.g., right triangles) and algebraically derive the hypothesis about how much larger the increased area will be. Different problem-solving strategies can then be compared among students, discussed, and even verified or tested on new triangles. Based on the experiences gained from these actions, students build their own knowledge about the problem under investigation (Antoliš et al., 2019).

Goldin and Shteingold (2001) cite various reasons for learning mathematics through problem solving and research: in problem solving and research, we connect various mathematical contents; we develop higher levels of thinking, higher cognitive processes; problem solving and research contribute to conceptual development and understanding of mathematical concepts; the teacher has an opportunity to gain insight into students' knowledge, understanding, and learning difficulties; through the process of problem solving, various strategies are developed (e.g., constructions, computational algorithms, data organization, tabulation, drawing diagrams, etc.); problem solving and research require decision-making and encourage student collaboration and discourse based on rational, logical analysis; it connects with other important (mathematical) ideas and encourages creativity.

Problem-solving and research enable students to learn content and process simultaneously. In addition to content, they learn how to solve problems, evaluate solutions, think critically (Monalisa, et al. 2019; Tohir & Abidin, 2018). However, one of the goals of mathematics education is also to stimulate students' interest in mathematics and motivation for exploration (Da Ponte, 2007). It is desirable that these hours are designed so that every student should be successful in them. The teacher sets a complex event, question, or problem. The student (O'Brien et al., 2011):

- formulates hypotheses to explain events or solve the problem,
- collects data to test hypotheses,
- draws conclusions, and
- reflects on the original problem and the thinking processes required to solve it.

During these lessons, students use mathematical knowledge or learn procedural skills in various activities. They are part of mathematics education in which students comprehensively process mathematical or non-mathematical problems, from setting investigation goals to making and presenting findings reports (O'Brien et al., 2011). The tasks differ from typical mathematical tasks in that the goal is not entirely clearly defined or represents a choice within the task itself. This means that we mainly solve open-ended problem situations, characterized by only the starting point of thinking being given, with the problem situation presented to the student as a challenge, and the goal of the task only roughly defined (Hannula, 2019). Because open problems do not predefine goals or solution paths, they provide opportunities for decision-making and learning various problem-solving strategies. More important than solving the problem itself in these tasks are procedural knowledge, understanding the problem situation, asking meaningful questions, and justifying findings.

Problem-based learning: In problem-based learning, students are presented with a real problem (O'Brien et al., 2011). This problem initiates exploration while collaborating to find solutions. The desired outcome is the development of knowledge that is applicable and flexible, rather than inefficient. Inefficient knowledge consists of information that is memorized but rarely used.

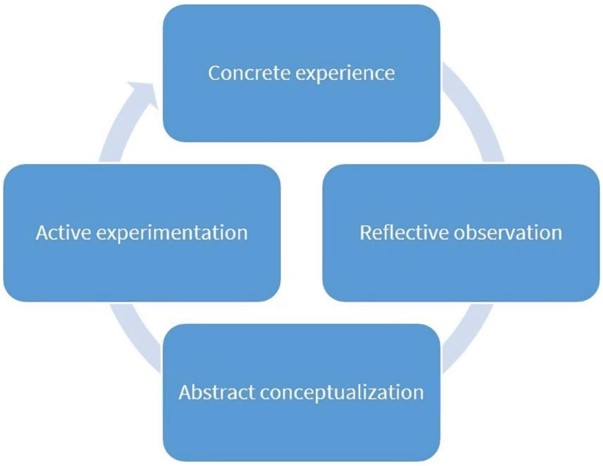

Figure 3. Schematization of Kolb's (1984) experiential learning cycle (Adapted from Kolb, 1984)

The goal of problem-based teaching is to learn the subject (mathematics) by seeking real solutions to problems. Problem-solving encourages thoughtful discussion, contribution of explanations and insights, and exploration.

Active learning: According to Fink (1999, 2003), active learning is based on logical reasoning and empirical verification, defined by two levels: the level of dialogue and the level of experience. Active learning promotes the development of mathematical thinking (creative, critical, analytical, and systematic; cf. Birgili, 2015). The level of experience includes (Fink 1999, 2003):

- Observation, which relates to how activities are connected to the topic under consideration (an opportunity for reflecting on knowledge),
- Activity, which involves any task, where the student independently performs or completes a task (e.g., presenting concepts with models, diagrams, searching for examples and counterexamples, experimenting).

Activities of active learning are diverse: modeling (finding a mathematical representation for a non-mathematical object or process; Freeman et al., 2014), seeking analogies (e.g., a triangle in a plane is analogous to a pyramid in space), providing/finding examples and counterexamples, independent solving of open problems (insight into problem situations and asking questions; Litster et al., 2020), experimenting, conducting measurements in the classroom or nature (Freeman et al., 2014), collecting data using models to reflect mathematical knowledge, presenting concepts with diagrams, models, drawings, estimation, independent source searching, finding similarities, differences, and connections between concepts and facts.

Social psychologists (Blaye et al., 1988) emphasize the importance of collaboration, as cognitive abilities develop in social interaction. Collaboration allows for confronting different viewpoints and insights into different ways of thinking, which can destabilize the original understanding of a concept or problem-solving procedure (Roschelle & Teasley, 1995). Because learning is not purely individual, students are not directed solely to their own thinking. Dialogue and collaboration enable the exchange of opinions, asking questions, justifying, discussing, checking findings, and confronting different ideas (cf. Baker, 2015). By working in this way, students are encouraged to cooperate, allowing them to articulate their ideas and solutions.

Experiential learning: Experiential learning is a method of learning that seeks to integrate direct experience (sensation), observation (perception), cognition (understanding), and action (behavior) into an inseparable whole (Illeris, 2007). It is not limited to conveying symbols: abstract knowledge of concepts and regularities but continually involves the experiences of participants in learning (Kolb, 1984; see **Figure 3**).

In order for the student to truly internalize newly acquired concepts and replace intuitive notions with them, it is necessary to provide them with various situations in which they can apply and confirm this new knowledge (Kayes, 2002). Experiential learning can help them gradually acquire and build conceptual representations, from concrete experiences to abstract concepts (Kolb & Kolb, 2005). It plays an important role especially for students whose starting point in learning is sensory perception, collection, measurement, and observation. We provide an example from the field of empirical probability: students engage in concrete activities, estimating, predicting, and empirically verifying (Kolb & Kolb, 2005).

Data processing (statistics, probability, & combinatorics)

At the end of the 20th century, mathematics curricula for elementary school first included topics of data processing and statistics (statistics, probability, combinatorics). The main reasons for introducing topics of data processing are, as follows:

- numeracy: tables, diagrams, surveys are part of our daily lives, e.g. newspapers, textbooks, computer-presented data, etc.;
- the need to know tools for communication: graphical representations and tables are regularly used in communication;
- the need for the ability to critically evaluate presented data: if we do not understand data visualization techniques and cannot critically evaluate them, we are very susceptible to manipulation (advertisements, elections, etc.);
- accessibility of computational tools for data processing.

In the early years of schooling, it is not yet "proper" learning of data processing, but rather children intuitively acquire their first knowledge only at a concrete level. This gradually prepares them for abstract understanding.

As a mathematical content, data processing in the early years of schooling certainly is not an end in itself. Among other things, it connects mathematics with other subjects and thus contributes to shaping the image of holistic teaching (Cotič & Hodnik, 1995). It broadens mathematical horizons, develops mathematical thinking, and encourages students' critical thinking about the world in which they live. These contents are not just technical in nature (understanding various data processing methods, systematic data presentation, etc.), but they are characterized by uncertainty and variability, allowing us to make decisions ourselves when exploring a multitude of uncertain and variable data (Cockcroft, 1982). Therefore, they provide students with the opportunity to decide on the procedures for discovering and solving problems by themselves. At the beginning of schooling, statistics is merely an introduction to data presentation and analysis; therefore, it represents an activity that is necessary in a world full of information. Thus, statistics content must be presented to students at the elementary level, gradually helping to develop a critical attitude toward "numerical information" presented by the media (Howson & Kahane, 1986).

Statistics also allows for connections with other subjects, such as environmental studies and the acquisition of graphic representation skills in both mathematics and other subjects. The primary goal of teaching combinatorics is to develop logical thinking and a systematic approach to solving combinatorial situations in students, as well as fostering autonomy in their thinking. Of course, at the beginning of schooling, we cannot talk about combinatorics as a mathematical discipline, but rather students solve simple combinatorial situations through direct experience (play) at a concrete level. It can be said that learning these contents reveals the "heart" of mathematics to students in later years of schooling (in secondary school) (combinatorial concepts are expressed using the language of set theory, results and methods from combinatorics are very useful and beneficial in other mathematical areas, especially in probability theory, etc.).

But what about probability content? Deterministic thinking alone is no longer sufficient to understand certain sciences; non-deterministic thinking schemes are increasingly necessary and present, for example, in genetics, biology, physics, economics, etc. (UNESCO, 1972). Probability is used today in areas close to everyday life: in meteorology (weather forecasts), in elections, in insurance, etc. Today's individual lives in a rapidly changing world and must increasingly face new and uncertain situations. Therefore, a "probability alphabet" is necessary, which requires a special way of thinking that is foreign to the deterministic mode of thinking predominant in our schools. Thus, even adults who have received such schooling often encounter difficulties in understanding basic and fundamental concepts of probability since binary logic (true/false, is/is not) fails here. Mathematics educationalist E. Fischbein conducted a study in Israel in 1970 on the suitability of introducing probability already at the primary level of mathematics teaching and concluded that the student at the primary level intuitively accepts and acquires the most elementary concepts of probability well (Fischbein, 1975). With the research, the author confirmed the hypothesis that concepts and techniques of probability and statistics should be introduced into mathematics teaching at the primary level, rather than at the secondary level or even in high school, when human thinking is more formed. If we want people to develop thinking that is significantly different from deterministic thinking schemes, we must start teaching statistics, combinatorics, and probability (data processing) at the level of concrete operations (seven-11 years).

From Present to Future

In the third phase of changes in mathematics education (from 1990 onwards), there is a strong emphasis on the essential need for an uninterrupted connection between the initial steps of students into the world of mathematics and the teaching of other disciplines and life itself. The need and value of teaching and learning mathematics lie not only in acquiring its content and methods but also in its essential role in the development of the holistic personality of the learner. At the same time, mathematics is a useful tool in the life of every person, both adult and child, and adolescent.

Mathematics literacy

Related to these insights, in the mid-20th century, the concept of mathematical literacy began to appear in curricular documents as an analogy to literacy, emphasizing proficiency in mathematical processes rather than mere knowledge of mathematical content (Jablonka, 2015). Somewhat later, especially in the context of adult education, the concept of numeracy emerged, initially meaning something akin to "scientific literacy." Over the following decades, both concepts acquired a different connotation: mathematical literacy today means the ability to use mathematics in one's professional work, social environment, and further education. This is also emphasized by various versions of the definition of mathematical literacy in *program for international student assessment* research. The definition from 2022 reads, for example:

> "Mathematical literacy is defined as the ability for mathematical reasoning and the transformation, application, and interpretation of mathematics to solve problems in various life situations. It includes concepts, procedures, facts, and tools for describing, explaining, and predicting phenomena. It helps individuals understand the role that mathematics plays in the world, helps them make reasoned conclusions and decisions needed as constructive, engaged, and thoughtful citizens of the 21st century" (OECD, 2023).

Mathematical literacy should not be equated with school mathematics (Gardiner, 2004), which is broader and more extensive at higher levels of education. A significant part of school mathematics, whether it concerns concepts, procedures, processes, or strategies, is useful in everyday or professional environments. Furthermore, the use of mathematics requires both certain mathematical knowledge and skills and processes. For effective use of one's mathematical knowledge, for example, it is important to be able to collect and process data, communicate and receive mathematical information, know appropriate technological tools, and so on. One of the most important demands of mathematical literacy in the 21st century is the ability for mathematical modeling.

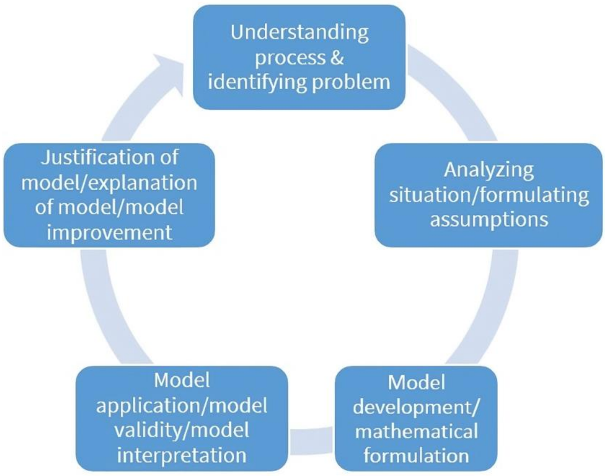

Figure 4. Process of modeling in mathematics (Source: Authors' own elaboration)

Table 2. Car values

Car age (in years)	Cost of car XXS (in 1,000 €)	Car age (in years)	Cost of car XXS (in 1,000 €)
1	9.0	10	3.5
2	7.7	11	2.4
3	5.6	12	2.4
4	4.9	13	2.2
5	3.7	14	2.0
6	4.7	15	2.1
7	3.4	16	2.0
8	3.3	17	1.9
9	2.8	18	1.9

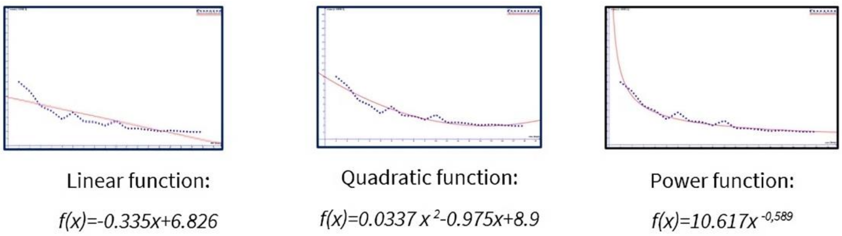

Figure 5. Interpolating functions (Source: Authors' own elaboration)

Modeling as part of mathematical literacy: Mathematical modeling is a process of mathematization. We introduce the principles and principles of mathematics into selected situations, thus translating reality into a mathematical environment. For example, population growth, which increases exponentially, is modeled by an exponential function. The mathematical model for projectile motion is a parabola. The decline in the value of goods in the market (e.g., with age, the values of cars, houses decrease, etc.) can be represented by exponential decay, and bacterial growth is modeled by exponential growth. Modeling assumes knowledge of the modeled phenomenon (e.g., physical laws), mathematical tools, modeling techniques, and criticality in the use of models (Žakelj, 2010). Mathematical modeling is a cyclical process consisting of several steps. The phases of the mathematical modeling process are shown in **Figure 4**.

We present an example of empirical modeling (**Table 2**).

Task: *We are interested in how the value of a car decreases over the years.*

Let us determine several examples of adjustment functions that can model the given data (e.g., linear, quadratic, and power functions) (**Figure 5**). Let us compare the models with each other based on how well the curve fits the points in the coordinate system. What conclusions can we draw about the value of the car over the years based on the data and adjustment functions? How could we verify the accuracy of the chosen model?

If we compare the model functions based on how well their graphs fit the empirically obtained data, we can say that all models are suitable within the given range, with the quadratic and power functions being slightly more accurate. To select the most appropriate model function, we must consider the context. Without considering the context (realistic situation), we cannot choose a more appropriate one between the last two.

Considering the context and the obtained data, we can conclude that the price drops the most in the early years, and later it decreases less and less. This can be observed by noting the differences in values in the early years (significant differences at the beginning, negligible at the end) or by observing the slopes of the graphs of the model functions (decreasing). We can also infer that the price of the car will never be zero, so the linear model is ruled out. Similarly, it is unlikely that the price would increase over the years (if we disregard the values of very old cars), so the quadratic function model is also ruled out. Thus, the most realistic model remains the power function: $f(x)= 10,617 \cdot x^{-0,589}$.

From a didactic perspective, the modeling process involves reflecting on mathematical knowledge. For example, when modeling a linear relationship between quantities, we reflect on the linear function; when modeling exponential decay, we reflect on the exponential function. Modeling also creates opportunities for learning to make assumptions and generalizations.

However, when transferring mathematical problem solutions to the real world, one must be aware of the limitations of the model, awareness of the conditions under which the model was created, that the formula may be only seemingly scientific, etc. In this context, a critical attitude towards evaluating results must be developed. Interpreting solutions from the perspective of a real situation is often very challenging, sometimes more so than interpreting them from a mathematical perspective.

Digitization in mathematics

In the 21st century, the overall learning process and mathematics instruction are also changing due to the increasingly common use of educational technology. Meaningful and purposeful use of educational technology in teaching can enhance learning (Clark-Wilson et al., 2020; Viberg et al., 2023). The use of digital technologies in mathematics instruction has two main functions (Clark-Wilson et al., 2020):

(1) supporting the organization of teacher's work (e.g., creating teaching materials, formative and summative assessment of students, etc.) and

(2) supporting new ways of teaching and presenting mathematics.

The use of digital technologies and resources in the learning process enables the creation of rich learning environments using various digital materials and digital supporting tools, applets, animations, and simulations; it supports various teaching approaches such as modeling, simulation, experimentation, and exploration, as well as solving mathematical problems and authentic tasks (Klančar et al., 2019). Furthermore, well-thought-out design and pedagogically appropriate use of digital technologies in the teaching and learning process of mathematics is crucial (Zbiek, 2003), especially from the perspective of developing mathematical intuition, understanding mathematical concepts, exploring relationships, creating accurate graphical representations, formulating and proving hypotheses, and using various problem-solving strategies, etc. However, many researchers also warn of the pitfalls and caution in the use of educational technology in teaching. It is sensible to consider that digital tools are used as a supplement to other teaching methods and not as a substitute, both in the planning of teacher training programs and in the implementation of programs (Hillmayr et al., 2020; Viberg et al., 2023).

Virtual reality & mathematics education: In recent years, there has been a notable surge in the widespread adoption and growing utilization of VR, particularly in educational contexts (Serin, 2020). VR technology enables users to fully immerse themselves in simulated environments through various equipment, allowing for interactive engagement and manipulation within these virtual settings. Its applications span a wide range of fields, including its promising potential in education.

Numerous studies have underscored the positive impacts of incorporating VR into education, particularly in terms of boosting student motivation, facilitating deeper learning, increasing time dedicated to learning tasks, and fostering enjoyment (Mystakidis et al., 2021; Sattar et al., 2020). Teachers also perceive VR technology as conducive to student engagement, visualization of complex scenarios, accelerated learning, and enhanced motivation and interest, as well as enjoyment (Alalwan et al., 2020; Serin, 2020).

However, the potential use of VR technology in mathematics education remains relatively understudied. Some researchers have proposed leveraging VR for real-life math tasks, allowing students to create mathematical models and integrate mathematics with real-world scenarios (Cakiroglu et al., 2023; Martín-Gutiérrez et al., 2017). Moreover, VR may aid students in grasping complex and abstract mathematical concepts through hands-on learning experiences (Martín-Gutiérrez et al. 2017), thereby enhancing instructional design and mathematical literacy (Cakiroglu et al., 2023). Cakiroglu et al. (2023) introduced an instructional design incorporating virtual tools and realistic mathematics education (VRME) to teach fractions, demonstrating its effectiveness in improving mathematics literacy skills among sixth-grade students. Additionally, VR could be utilized for multiview projections of 3D objects, allowing students to manipulate and visualize objects from different viewpoints, thus enhancing visuospatial skills (Dilling & Sommer, 2022).

Despite the positive effects of VR on mathematics learning, challenges such as insufficient teaching resources, limited interactivity, and technical issues persist (Liu et al., 2018). Therefore, it is crucial for educators to adequately plan teaching activities and policymakers should support prospective teachers in integrating VR technology into classrooms by providing training and resources.

Artificial intelligence & mathematics education: The integration of AI in mathematics education marks a significant advancement in pedagogy, offering novel opportunities to enhance learning experiences. AI algorithms can tailor educational content to individual student needs, providing personalized learning pathways that adapt to each learner's pace and style (Chen et al., 2020). This adaptive approach fosters a more inclusive learning environment, where students can receive targeted support and challenge (Salas-Pilco et al., 2022), ultimately promoting deeper engagement and understanding of mathematical concepts. Moreover, AI-powered educational platforms can generate real-time feedback, pinpointing areas of difficulty and offering customized recommendations for improvement (Xu et al., 2023). By analyzing student interactions and learning patterns, AI algorithms can identify misconceptions and provide targeted interventions, facilitating more effective remediation strategies.

Furthermore, AI technologies offer innovative tools for exploring mathematical concepts through interactive simulations and problem-solving environments (Hwang & Tu, 2021). Virtual tutors powered by AI can engage students in dialogue, answering questions, and providing explanations in real-time. These intelligent systems can scaffold learning experiences, guiding students through complex problem-solving tasks while fostering critical thinking and metacognitive skills. Additionally, AI-driven educational games and applications can transform abstract mathematical concepts into interactive and immersive experiences, making learning more engaging and accessible to diverse learners. Overall, the integration of AI in mathematics education holds immense potential to revolutionize teaching practices and empower students with the skills and competencies needed to thrive in the digital age.

CONCLUSIONS & FUTURE DIRECTIONS

The evolution of mathematics education over the past emphasizes the crucial role of mathematics in shaping holistic learner development and its practical significance in everyday life. The concepts of mathematical literacy and numeracy have emerged as pivotal frameworks, highlighting the importance of not only mastering mathematical content but also leveraging mathematical processes and reasoning in various real-world contexts. As mathematics continues to evolve, it becomes increasingly essential to foster mathematical modeling skills, enabling individuals to apply mathematical principles to analyze and solve complex problems effectively. Moreover, the integration of digital technologies in mathematics education presents new avenues for enhancing teaching and learning experiences, offering personalized learning pathways and immersive learning environments that cater to diverse learner needs.

Looking ahead, future research should focus on exploring innovative approaches to leverage emerging technologies, such as VR and AI, to further enhance mathematics education. VR holds promise for creating immersive learning experiences that facilitate deeper engagement and understanding of abstract mathematical concepts, while AI-powered educational platforms offer personalized learning experiences tailored to individual learner needs. Additionally, research efforts should aim to address challenges associated with the integration of these technologies, such as limited access to resources and technical issues, to ensure equitable access to high-quality mathematics education for all learners. By harnessing the potential of VR and AI technologies, mathematics educators can create dynamic learning environments that foster critical thinking, problem-solving skills, and mathematical literacy, preparing students for success in the digital age.

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