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How argumentation relates to formal proof process in geometry

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ABSTRACT

This study aims to examine how being involved in an argumentation process relates to the formal proof process in geometry. Prospective mathematics teachers were involved in an argumentation process while producing conjectures before engaging in formal proof of the recently produced conjectures. To collect data, four geometry-proof tasks that involve two sections were employed. The first section of the tasks demands the production of conjectures, which stands for the term argumentation. The second section asks for the formal proof of one of the recently produced conjectures. Based on the data analysis, the affordances of being involved in argumentation before engaging in the formal proof process were listed as positive affective occasions, arrangement of knowledge related to the content of the task, visual aspect, and the veracity of the statement. Negative affective occasions and confusion related to the difference between conjecturing and proving were coded as constraints of being involved in argumentation before formal proof.

Keywords: argumentation, proof, geometry

INTRODUCTION

Studies conducted with regard to mathematical proof have gained a new impetus in recent decades (Stylianides et al., 2016). According to Ellis et al. (2012), the proof is “part and parcel of doing mathematics and should be regular and ongoing part of learning of mathematics” (p. 8). By dealing with proving tasks, students may be provided with the opportunity to develop deep learning of mathematics. As argumentation is associated with proof, it is one of the concepts taken into consideration in research related to mathematics education from various aspects (Pedemonte, 2007).

Pedemonte (2002) offered an analogy related to argumentation and proof; “the processes used to construct a conjecture and its validation: argumentation and proof” (p. 71). Garuti et al. (1996) introduced this perspective regarding argumentation and proof using the term “cognitive unity.” Cognitive unity emphasizes the possible use of the elements of the production process of conjectures during the proving process (Garuti et al., 1996). When cognitive unity is established, it refers to the fact that the argumentation section involving conjecturing reinforces the proving section (Pedemonte, 2007). Garuti et al. (1998) also mentioned the gap between the process in which the statement is explored and the process in which the related statement is proved. Whilst Garuti et al. (1998) used the term “gap,” Fiallo and Gutiérrez (2017) used the term “cognitive rupture” for the same issue. In a more general manner, Antonini and Mariotti (2008) considered that cognitive unity, as a theoretical construct, covers the possibility of a relationship and rupture between argumentation and proof.

In this study, the presence of both cognitive unity and cognitive rupture was taken into consideration. For this purpose, it was aimed to investigate both the affordances and constraints of being involved in an argumentation process before engaging in the formal proof process. By avoiding the constraints and employing the affordances during the instruction, the formal proof processes of prospective mathematics teachers might be enhanced. In this manner, the main purpose of the present study is to investigate how prospective mathematics teachers’ involvement in an argumentation process relates to their formal proof processes in geometry. In more detail, the concept of formal proof refers to “a specific type of argument, which is a connected sequence of deductive, logical statements in support of or against a mathematical claim” (Ellis et al., 2012, p. 22). In this study, formal proof stands for stating appropriate methods, definitions, and mathematical language by following logical steps to verify the conjectures.

This study is derived from the dissertation of Esra Demiray conducted under the supervision of Mine Isiksal-Bostan and Elif Saygi.

Argumentation and Proof

Different conceptions related to argumentation and proof have led to the multiplicity of approaches followed in studies. The approaches related to argumentation and proof can be subsumed under three categories. Firstly, some studies did not point out a clear-cut difference between argumentation and proof (Pedemonte, 2007). For example, Pedemonte (2007) presented the phrase, "empirical proof," which was mentioned in some studies (e.g., Fiallo & Gutiérrez, 2017; Harel & Sowder, 2009) as an example of cases, where word "proof" was also employed for both deductive and empirical cases. Moreover, the presence of studies investigating not only the argumentation process of the generation of a conjecture but also the argumentation process of its proof presents the issue that proving can be considered an argumentation process in its own right (Pedemonte, 2007).

In the second approach, some studies referred to the presence of a distance between argumentation and proof (e.g., Antonini & Mariotti, 2008; Fiallo & Gutiérrez, 2017; Garuti et al., 1996; Pedemonte, 2002, 2007) with the help of the translations of the studies conducted by Duval (1991, 1995). Duval (1995, as cited in Pedemonte, 2007) asserted the existence of a structural gap between argumentation and proof although a similar syntactic is used in both phases. As mentioned, some studies labeled the referred gap between argumentation and proof as cognitive rupture (Antonini & Mariotti, 2008; Fiallo & Gutiérrez, 2017). Similarly, the aforementioned issue was expressed through the distinction or the difference between argumentative reasoning required to produce the conjecture and deductive reasoning needed to validate it (Pedemonte, 2007). More precisely, it was stated that argumentation is composed of any rhetoric case deployed to convince and show whether the statement is true or false. On the other hand, the proof was described in a more formal manner by referring to the theoretical validation of a statement followed by logical concatenations (Duval, 1995, as cited in Antonini & Mariotti, 2008).

The last approach emphasizes the existence of continuity between argumentation, which refers to the process of conjecture production distinctively and the proving process of the recently produced conjecture. In other words, the issue focused on is the relationship between the conjecturing phase and proving (Pedemonte, 2007). As stated, Garuti et al. (1996) introduced the concept of cognitive unity of theorems. Cognitive unity, which dates back nearly twenty years, was offered to determine the proper conditions in terms of providing a smooth approach to proving (Boero, 2017). Garuti et al. (1998) defined the cognitive unity of a theorem as "the continuity existing between the production of a conjecture and the possible construction of its proof" (p. 345). The pioneering studies related to cognitive unity have presented evidence for the case that when students are involved in an argumentation process to produce a conjecture, the proof of the recently stated conjecture becomes more accessible to them (Garuti et al., 1996, 1998). Moreover, it was seen that the personal arguments of students in the statement production stage were kept in the same way in the proving stage, and they were apt to maintain the type of reasoning, use similar expressions, and pursue similar steps (Garuti et al., 1996).

The approaches regarding argumentation and proof mentioned above are mainly related to cognitive factors. In a mathematical task involving argumentation and proof, cognitive and affective aspects are critical and intertwined (Furinghetti & Morselli, 2009; Selden & Selden, 2013). Although a growing number of studies indicate that students at all levels have cognitive difficulties in argumentation and proof, these difficulties might partially originate from affective factors (Furinghetti & Morselli, 2009). In addition, affective factors might play a positive role in the proving process (Selden & Selden, 2013). Due to the scarcity of studies, which examine the affective side of proof and the connections between affective and cognitive factors in the proving process, further studies are called to focus on these issues (Selden & Selden, 2013). In this study, the affective aspects related to argumentation and proof were also taken into consideration.

Conceptual Framework

To analyze cognitive unity and rupture between argumentation and proof, Fiallo and Gutiérrez (2017) deployed the categorization of Pedemonte (2005), which includes structural analysis and referential analysis. The structural analysis aims to examine whether the statements employed in the argumentation are also used in the proof. In such a case, it is called structural cognitive unity. The absence of such a link between the statements is called structural cognitive rupture (Pedemonte, 2005, as cited in Fiallo & Gutiérrez, 2017). Moreover, if the ways of inference have the same structure, such as deduction, induction, and abduction in both argumentation and proof, it presents the structural continuity between them. For example, if some abductive steps of argumentation are used in proving, it refers to structural continuity (Boero et al., 2010). In referential analysis, it is scrutinized whether the systems of signs such as drawings and notations and the systems of knowledge such as theorems and definitions are used in both argumentation and proof processes. When used in both processes, the presence of referential cognitive unity is stated; otherwise, it refers to referential cognitive rupture (Boero et al., 2010; Pedemonte, 2005, as cited in Fiallo & Gutiérrez, 2017).

As a further step, Fiallo and Gutiérrez (2017) combined the mentioned categorization with the cK¢ model presented by Balacheff and Margolinas (2005), the argumentation model of Toulmin (2003), and the types of proof proposed by Marrades and Gutiérrez (2001). By doing so, Fiallo and Gutiérrez (2017) unfolded four categories of cognitive unity and rupture between argumentation and proof: empirical cognitive unity, deductive cognitive unity, referential rupture and empirical structural unity, and referential unity and structural rupture. For example, in empirical cognitive unity, there is both empirical referential unity and empirical structural unity. There is empirical referential unity when the group uses some notions, such as backing, representation, operators, and controls in both argumentation and proving. There is empirical structural unity when they also have an empirical stance while explaining conjecture and proving attempts. As another example, in referential rupture and empirical structural unity, the group works empirically in both phases so that the presence of empirical structural unity is stated. Referential rupture is added when the group uses new operators and controls while trying to present a deductive proof although the product is still a generic proof (Fiallo & Gutiérrez, 2017).

While framing such an analysis structure, the aim of Fiallo and Gutiérrez (2017) was to investigate the concepts of cognitive unity and rupture by approaching them from two aspects. The first aspect was to examine how the presence or absence of a connection between the process of conjecture production and the process of its proof affects the type and structure of proof, such as deductive or empirical. The second aspect was to inspect whether the types of proofs progress by means of the intervention conducted in the study. Based on the study of Fiallo and Gutiérrez (2017), this study basically focuses on the details of the mentioned four categories of cognitive unity and rupture in terms of investigating whether the case is an affordance or constraint for the formal proof process. In other words, the mentioned categories guided the development of the research question of this study.

Proof in Mathematics Education

The significance of proof in mathematics education is underlined in the research and paid attention by involving them in the mathematics curricula of different countries around the world. However, there is a widespread result in the research related to proof that not only students at different levels but also mathematics teachers have difficulty in conducting proof involved tasks (Ellis et al., 2012; Mejia-Ramos & Inglis, 2009; Reiss et al., 2008; Stylianides & Stylianides, 2017). Even mathematicians who worked on the proving process of many theorems might struggle in conducting the proof of a new statement and spend plenty of time to complete. Thus, it is an expected situation that students have difficulty in such a demanding task throughout school mathematics (Ellis et al., 2012).

Although many suggestions to overcome these difficulties in proof were depicted in the studies, the persistent struggles of students were an inevitable part of the mentioned research area. By drawing on the findings of the study conducted with undergraduate students in both mathematics and mathematics education, Moore (1994) underlined some major difficulties, which are deficiency in the formation and applications of definitions of mathematical concepts, giving proper examples, inadequate concept image and intuitive understanding regarding the concepts, accurate use of the notation and language in mathematics, and structuring the beginning of a proof. Research in the literature presented that some secondary and undergraduate students accept empirical arguments such as examples and measurements as qualified enough and convincing while validating or trying to prove a statement (Jahnke, 2007). This situation might have originated from the fact that students could not completely understand the epistemological meaning of the concept of proof (Jahnke, 2007). Another factor causing students to have difficulty in proof might be the discrepancy between the pragmatic and cognitive stance of argumentation and the theoretical nature of proof (Antonini & Mariotti, 2008). However, it was strongly suggested that the focus of the proof related research should be shifted from the ones, which document the challenges of students while engaging in proving to the ones, which offer the methods to overcome the problems such as the suggestions related to the classroom-based intervention (Stylianides & Stylianides, 2018).

To help students to experience deductive reasoning during teaching is a difficult issue (Jones, 2002). It was underlined that "there are a relatively small number of research studies that have developed promising classroom-based interventions to address important issues of the teaching and learning of proof" (Stylianides et al. 2017, p. 258). In this respect, cognitive unity was presented "to bring about a smooth approach to theorems in school" (Garuti et al., 1998, p. 345). Both cognitive unity and rupture might be taken into consideration to inspect proof at the undergraduate level within the context of geometry. To present both affordances and constraints of being involved in an argumentation before formal proof might be a starting point to offer effective methods to teach proof related concepts.

The Rationale of the Study

From a broader perspective, proof process involves the collocation of a series of tasks, which are "experience, insight, reasons, constructions, and arguments, gained through an exploration of a specific situation" (Sinclair et al., 2012, p. 48). Similarly, according to Ellis et al. (2012), the proving process is a broad construct; hence, it is attributed to various tasks leading to writing a proof such as producing conjectures, looking for generalizations, considering examples and counterexamples, and searching for similarities among cases. Nevertheless, it is not usually the case for students since they are not the attendants of the whole process of proving mentioned above. That is, students directly enter the proving process before arousing their curiosity and giving them the opportunity to carry out inquiry related to the concept in question (Sinclair et al., 2012). In addition, students are generally asked to understand and then replicate the proofs of the statements, which are not produced by students, in a way that is presented by the instructors. The tasks through which students deal with the proof of a statement they have produced by means of an argumentation process offer a significant potential to improve their ability to prove (Garuti et al., 1996). As Baccaglini-Frank (2010) stated, "passing from the development of a conjecture to the construction of a proof is a delicate process" (p. 3). This study emphasizes the importance of the mentioned elements of a proof-aimed process. To develop students' competencies in the mentioned elements of proving, mathematics teachers have a critical role. Mathematics teachers are responsible for raising students' awareness and promoting their ideas with regard to the necessity of proof (Stylianides et al., 2016). Thus, prospective mathematics teachers should be provided with the opportunity to experience the mentioned components of proving. It should be determined how effectively argumentation and proof can be integrated into undergraduate courses of prospective mathematics teachers. In this respect, it is important to investigate how the argumentation process of prospective mathematics teachers supports or hinders the following formal proof process.

Proving can be used effectively in mathematics teaching when appropriately set regardless of the domain (Ellis et al., 2012). In this study, the selected context related to argumentation and proof is geometry since geometry is "a rich source of opportunities for developing notions of proof" (Jones, 2002, p. 125). Moreover, geometry is a proper area to develop the reasoning and justification skills of students (National Council of Teachers of Mathematics [NCTM], 2000) and provides the opportunity for conjecture production and exploration (Gillis, 2005). To prepare the occasions in which students might move between the theoretical and practical perspectives of geometry is a critical challenge in terms of mathematics education (Fujita et al., 2010). Furthermore, students should be offered environments, where they can see the differences between experimental and deductive approaches in the geometry domain. In this manner, this study aims to contribute to the literature by pointing out how the argumentation process should be arranged to support the following proving process in geometry.

To present the affordances and constraints of experiencing argumentation before the formal proof is important in terms of education since it might be helpful to meet students' needs while learning geometry. For example, teacher educators might consider the results of this study while teaching proof-related concepts in undergraduate geometry courses and helping prospective teachers to integrate core geometry with geometry teaching. In addition to geometry related courses, the inferences of this study could be integrated into other courses in mathematics teacher education programs. Moreover, to overcome the undergraduate students' difficulties in proof due to the quick transition to formal proof in a variety of domains in mathematical courses, it was suggested that a transition to proof course at the beginning of the undergraduate program could be offered to prepare a bridge through formal proof and to help them to use mathematical language effectively (Moore, 1994). In terms of the present study, the affordances and constraints of being involved in an argumentation before the formal proof process could be considered to facilitate students' learning in such kind of transition to proof courses.

All in all, to find out the affordances and constraints of the argumentation process with respect to proof might be helpful to enhance the methods used while teaching proof. In the light of these issues, the research question, which guided this study was stated as follows: How does prospective mathematics teachers' involvement in an argumentation process while producing conjectures relate to their formal proof process in the context of geometry?

METHOD

Research Design and Participants

Although the related literature presents information regarding the main issues of this study, which are the relations between argumentation and proof, a detailed investigation by considering different aspects such as affordances and constraints is aimed in this study. Since qualitative research aims to enhance a multifaceted and in-depth understanding regarding the central phenomenon of the study and to explore the contexts in which the related problems or issues emerge (Bogdan & Biklen, 2007), qualitative research was utilized in this study. Since phenomenology focuses on "describing the 'essence' of a phenomenon" (Merriam, 2009, p. 87), phenomenology was determined as research design. In more detail, phenomenon of this study is how argumentation relates to proof in geometry.

The present study was conducted with six junior prospective mathematics teachers (PMTs) in Ankara, Turkey. The participants were selected by employing purposeful sampling since it was aimed to determine "information-rich cases for study in depth" (Patton, 2002, p. 46). Based on the pilot study, it was decided to conduct the main study with PMTs with the highest grades in geometry and proof courses and the highest cumulative grade point averages. At the time of data collection of the main study, there were nearly 70 juniors in the program. Since proving is already a difficult and challenging task, the upmost six PMTs were selected to be able to examine rich processes. In addition, the participants worked in groups of three during geometry-proof tasks. To organize an environment, where PMTs could work effectively, they were asked to determine three members for each group. To diversify the argumentation process, which is the first section of geometry-proof tasks, the first group (G1) was allowed to use compass-straightedge, and the second group (G2) was allowed to use GeoGebra. All participants were female and had GPAs ranging from 3.30 to 3.55 out of 4.00.

The undergraduate teacher education program in question involves mathematics courses particularly related to proof and geometry such as discrete mathematics and geometry in the first year of the program. There are also some other mathematics courses, which do not involve chapters dedicated to proof but cover various types of proofs in other mathematical domains such as calculus and linear algebra. Moreover, the graduates of this program can be recruited as mathematics teachers in the 5^{th}-8^{th} grades in Turkey.

Data Collection

The participants have given their voluntary consent to participate in the study. Tasks based on geometry and proof were aimed to be used to collect data. In this respect, each task was planned to comprise a conjecture production phase by working on geometric constructions and a sequent proving phase. Initially, twelve tasks were prepared by scanning the geometry and proof-related literature (e.g., Alexander & Koeberlein, 2011; Gutenmacher & Vasilyev, 2004; Morris & Morris, 2009; Velleman, 2006). However, based on the opinions of experts and the pilot study, four of these tasks, which were thought to be the most appropriate in terms of the logic of cognitive unity and rupture, were determined and administered in the main study to collect data.

At the first step of the data collection process, the groups were informed about argumentation, reasoning, and proof in mathematics education, methods of proof, proof schemes, notation and formal writing in proofs, proof in geometry, and examples of proof regarding geometry concepts. After this teaching week, four geometry-proof tasks were administered. Each task involved two worksheets; the groups were expected to produce conjectures by virtue of geometric constructions while working on worksheet A, and they were expected to prove one of the conjectures they had recently produced on worksheet B. A list of possible conjectures directly related to each task was prepared before the application of the tasks. Of the conjectures the groups produced during the argumentation, one conjecture was determined per task for the proof section. The conjectures presented in worksheets B are listed in **Table 1**.

Table 1. Conjectures, which were asked to be proven

Tasks	Conjectures
Task 1 (T1)	The perpendicular bisectors of the sides of a triangle are concurrent, and this point is the circumcenter of the triangle.
Task 2 (T2)	The altitudes of a triangle are concurrent.
Task 3 (T3)	The orthocenter, the circumcenter, and the centroid of a triangle are collinear.
Task 4 (T4)	Suppose that the points X, Y, and Z are placed at random on the sides of ∆ABC. Then, three circles, each of which is passing through one vertex and two points marked on the adjacent sides, are drawn. Three circles (AXZ, BXY, and CYZ) are concurrent.

All in all, among the conjectures, the most general ones were selected to ask for proof so that all groups could work on the proof of the same statements at the end of the tasks. There was not a time limitation while the groups were working on geometry-proof tasks. It was seen that the groups worked on each task for nearly three hours. According to Brown (2017), collective argumentation enhances the quality of mathematics education, helps students to deal with higher-order thinking skills, and encourages productive talks in the classroom. Thus, the groups were asked to work on the tasks as a team in a face-to-face setting and think out loud to enhance collective argumentation.

Data Analysis

Data were collected through documents and video recordings of the groups during geometry-proof tasks. Content analysis might cover a deductive approach or an inductive approach. When the studies in the literature cannot provide adequate knowledge about the phenomenon that is focused on and it is needed to derive the codes from the collected data, the inductive approach can be utilized (Elo & Kyngäs, 2008). It can be stated that it is the case for the analysis of the data that addresses the research question. To this end, the study followed the stages of inductive content analysis, which are "open coding, creating categories and abstraction" (Elo & Kyngäs, 2008, p. 109). In each geometry-proof task, the argumentation and proof processes were examined in detail and compared, and then the related codes and themes were arranged.

Trustworthiness of the Study

The descriptions of validity and reliability in quantitative research might not be adequate and completely applicable to the perspective of qualitative research (Golafshani, 2003). In this respect, Guba (1981) and Lincoln and Guba (1985) focused on the trustworthiness of the study and offered to use the terms credibility, transferability, dependability, and confirmability instead of internal validity, external validity, reliability, and objectivity, respectively.

To support the credibility of the study, triangulation and peer examination (Merriam, 2009) were employed. By collecting the data via different sources such as the video recordings of the activities, documents, and GeoGebra files, it was aimed to learn more about the phenomena focused and increase the credibility. In terms of peer examination, a doctorate student in mathematics education was the second coder in the data analysis process. The inconsistent points in the analyses of two coders were discussed and the analyses were continued until a consensus was reached. Similarly, to increase dependability, triangulation and peer examination (Merriam, 2009) were used. Since the generalization is not aimed at qualitative studies (Merriam, 2009), it was suggested to present a rich description to enhance the transferability to other settings (Lincoln & Guba, 1985). To ensure transferability, a detailed description of the study was presented. The context and participants of the study, data collection procedure, and data analysis were explained in detail. Finally, to ensure confirmability, triangulation and detailed description were utilized. Besides, to support confirmability, the excerpts taken from the recordings of the tasks were presented.

RESULTS

Both affordances and constraints of being involved in an argumentation while producing conjectures before engaging in a formal proof process within the context of geometry are presented below.

The affordances of being involved in an argumentation process before proving might be examined via four codes, which are positive affective occasions, arrangement of knowledge related to the content of the task, the visual aspect, and the veracity of the statement. On the other hand, the constraints of participating in an argumentation process before proving might be inspected via two codes, which are negative affective occasions and confusion related to the difference between conjecturing and proving. Some examples were given as follows to describe the aspects and the related codes in **Table 2**.

Table 2. Affordances & constraints of argumentation before formal proof

	Codes
Affordances of argumentation process before formal proof process	Positive affective occasions
	Arrangement of knowledge related to the content of the task
	Visual aspect
	Veracity of the statement
Constraints of argumentation process before formal proof process	Negative affective occasions
	Confusion related to the difference between conjecturing and proving

Affordances of Argumentation Process before Formal Proof Process

The first issue mentioned in the affordances of the argumentation process before the formal proof process is the positive affective occasions. It involves the cases, where PMTs increased their motivation when they offered a working idea, had a positive attitude toward the whole task, and learned to work collaboratively. To exemplify the mentioned positive affective occasions, an excerpt is given below.

This example is related to the increased motivation of PMT4 and PMT5, who are the members of G2, towards conducting the whole task when they offered effective methods and ideas in task 1 (T1).

T1 (during conjecturing)

PMT4: I hope, they intersect (*she was working on the construction of the perpendicular bisectors, and she hoped that their intersection would work in terms of the aimed construction*).

…

PMT4: It worked, it worked (*she was calling for the attention of others in G2*).

…

PMT4: I want to find something else (*she referred to finding another approach for performing the aimed construction*).

T1 (at the beginning of proving)

PMT4: I think, we are proving now what we have done recently.

PMT5: Exactly.

PMT4: If we can prove, we can answer why we said that. Let's do (*she referred to conducting proof*).

As seen above, the first occasion is from the period that PMT4 was working on the construction of the circumcircle by using GeoGebra. They tried some approaches such as finding the intersection of the angle bisectors, but they could not come up with a correct approach. Then, PMT4 noticed that they did not try the "perpendicular bisector" tool and started to work on it by hoping it would work. As deduced from the video recordings, she was quite excited when her idea worked, and she immediately explained it to the others in the group. After working on the mentioned approach collectively, she kept her enthusiasm and declared that she also wanted to find other methods. Then, at the beginning of the proving section of G2 in T1, PMT5 stated that they were asked to prove what they have reached recently. Similar to the first section of the task, PMT4 was also the kick-start of the group in the proving section. Aligned with the video recordings, it can be stated that she presented her high motivation for proving by uttering that "let's do". Moreover, it can be inferred that she might feel more confident in proving since they were asked to prove what they had found recently.

As another example of the positive affective occasions, some sentences of PMT2, who is a member of G1, in task 2 (T2) are given below.

T2 (during conjecturing)

PMT2: I think, this is a very logical method (*she referred to the approach she used while constructing the altitudes of the triangle. It is the basic construction used to draw the line that is perpendicular to a given line from a point not on the given line*).

…

PMT2: I have done it so perfectly that we should present it as a good example, I think (*she was proud of what she conducted with the mentioned approach*).

T2 (while working on proving)

PMT2: If I can show it, this would be like a complete proof (*she was trying to prove the statement that the altitudes of a triangle are concurrent by trying to set up congruence by means of angles. At this point, she is trying to show the equality of two angles*).

…

PMT2: I found many fine equalities and equations. I am trying to connect them right now.

The first excerpt given above is from the construction section in T2, and it presents how PMT2 was motivated since she offered a working idea and improved her self-efficacy regarding the whole activity. In more detail, the first part of the excerpt involves the sentences of PMT2 from the geometric construction attempt in T2 in which the construction of the altitudes and the orthocenters (if they exist) of the given triangles were asked. While constructing the altitudes of the given triangles, PMT2 used one of the basic construction approaches. That is, she regarded the construction of the altitude of a triangle as the construction of the line that is perpendicular to a given line from a point not on the given line. As seen, in the first sentence, she appreciated the logic underlying this approach. Moreover, through the second sentence given, it can be stated that she presented her pleasure for finding the approach, which increased her self-efficacy regarding the whole task. The second part of the excerpt given above is from the proving section of the same task. As seen from the video recordings, PMT2 did not stop working on proving attempts. Moreover, as deduced from the sentences, she kept her beliefs regarding the case that their argument would reach a valid proof and did not step back.

The second code given in the affordances of argumentation before proving is the arrangement of knowledge related to the content of the task. More precisely, it covers the following cases: PMTs evoked their previous knowledge through the general and auxiliary ideas in the conjecture production phase, consolidated their knowledge based on the conjectures they produced, promoted their geometric knowledge with discussion, prompted the knowledge about geometric concepts from previous tasks, discussed the proper usage of notations, terms, and expressions during argumentation, and clarified the issues before proof.

The following excerpt, which was taken from the argumentation of G1 during the conjecturing phase of task 3 (T3), serves as an example of the arrangement of knowledge code.

> T3 (while working on the construction of the centroid and orthocenter)
>
> PMT1: We found the centroid before, did not we? Is it the intersection of the angle bisectors? (*actually, they did not work on the construction of the centroid before by means of the tasks*).
>
> PMT2: Yes, it (*the centroid*) is the intersection of the angle bisectors.
>
> PMT3: The angle bisectors or the medians? (*she pointed out her hesitation*).
>
> PMT2: It (*the centroid*) is not related to the medians.
>
> PMT1: We used the medians while constructing the orthocenter, did not we?
>
> PMT3: I think, it (*the median*) is for the centroid.
>
> PMT2: The angle bisectors or the medians? Hmm (*she agreed on the hesitation*).
>
> PMT1: Let's try.
>
> …
>
> PMT1: The intersection of the angle bisectors is not the centroid.
>
> PMT2: Then, it (*the centroid*) is the intersection of the medians.
>
> …
>
> PMT1: We used the perpendicular bisectors of the sides (*of the triangle*) to find the orthocenter, but then we noticed imm…
>
> PMT2: We noticed that what we have found is not the orthocenter.
>
> …
>
> PMT2: The orthocenter is not the intersection of the perpendicular bisectors of the sides.
>
> PMT3: It is the altitudes (*she referred to the fact that the intersection of the altitudes of a triangle is the orthocenter*).

The given dialogue parts emerged while G1 was working on the construction of the centroid and the orthocenter of a given triangle. Unfortunately, it shows the fact that PMTs had some deficiencies in even basic concepts of geometry. They could not be sure of the meaning of the centroid and orthocenter of a triangle and how they could construct them for a while. In more detail, PMT1 and PMT2 thought that the centroid of the triangle could be constructed by finding the intersection point of the angle bisectors. However, PMT3 hesitated with this idea and offered that the centroid could be constructed by finding the intersection of the medians. Then, they decided to try each idea. The geometric figures in these attempts are presented in **Figure 1**.

In the first triangle of **Figure 1**, G1 constructed the intersection of the angle bisectors (point O) and the intersection of the medians (point G). After trying constructions, they concluded that they should work on the medians to construct the centroid. In the second triangle, G1 also constructed the intersection of the perpendicular bisectors of the sides (point D). Then, they also concluded that the orthocenter is the intersection of the altitudes of a triangle, not the intersection of the perpendicular bisectors of the sides. All in all, after an exploration and trial-and-error process, they noticed their incorrect interpretation and reached the correct knowledge regarding the mentioned points. Thus, G1 evoked and consolidated the knowledge related to the mentioned two points before passing to proof. The argumentation process before proof helped them to arrange the geometry knowledge related to the task.

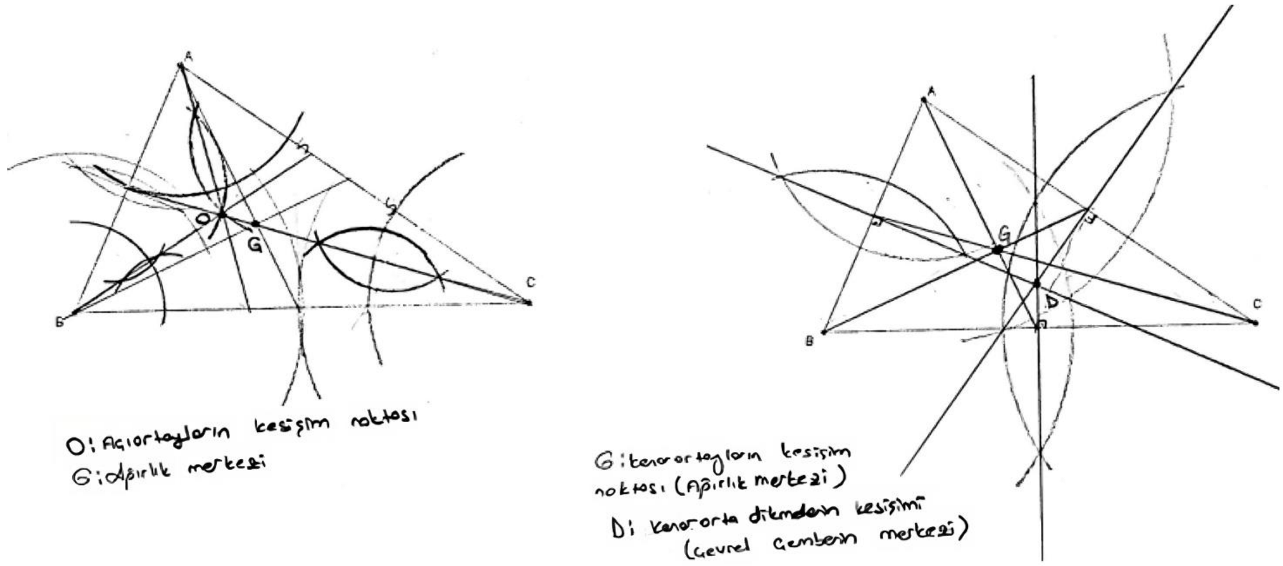


Figure 1. Geometric construction attempts of G1 in T3 (Source: participants' own illustrations, reprinted with their permission)

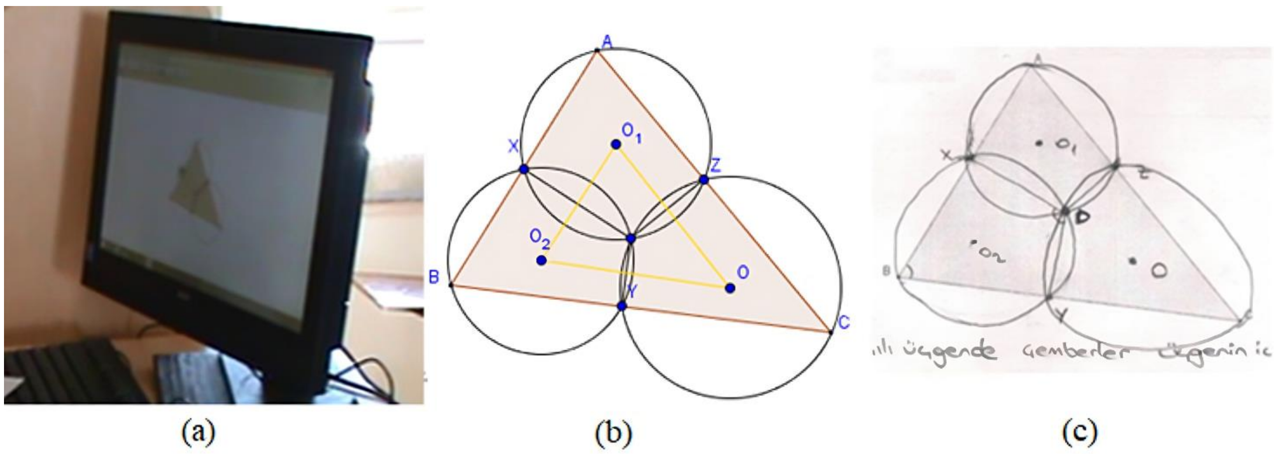


Figure 2. Cyclic quadrilaterals that G2 formed during geometric construction in T4 (Source: (a) screenshot from video recordings of participants, (b and c) participants' own illustrations, reprinted with their permission)

The third issue given in **Table 2** related to affordances is the visual aspect. Since the groups worked on geometric constructions preceded by proof, this process increased their awareness of the geometric objects visually and helped them to notice some key drawings, which might be used during proving. Moreover, being involved in such an argumentation led the participants to present the properly formed geometric figures while writing the related proofs. The following dialog serves as evidence for the visual aspect. It has parts from both conjecture production and proving sections of G2 in task 4 (T4).

T4 (during conjecturing)

PMT5: I will open what we have done *(she was finding the GeoGebra file they worked on)*.

PMT4: Look, for example, A, X, this point *(point Y in b of **Figure 2**)*, and this *(point D in c of **Figure 2**)* constitute a cyclic quadrilateral.

…

PMT4: For example, the sum of this angle and this angle is 180 *(she referred to the case that the sum of ∠XBY and ∠XDY given in c of **Figure 2** is 180°)*.

PMT5: Which angles?

PMT4: According to the property of cyclic quadrilateral, what is the sum of these angles? *(the sum of ∠XBY and ∠XDY given in c of **Figure 2**)*.

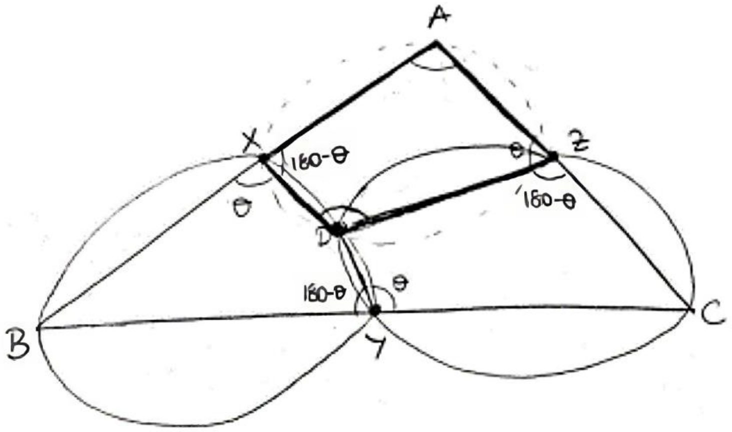


Figure 3. Cyclic quadrilaterals G2 formed during proof in T4 (Source: participants' own illustrations, reprinted with their permission)

PMT5: The sum of these two *(angles)* is 180.

…

PMT5: Yes, it reached 180 *(she calculated the sum of the mentioned angles by using GeoGebra)*.

…

PMT6: One minute. This is our figure, there is one cyclic quadrilateral here *(the cyclic quadrilateral XBYD in c of **Figure 2**)*, one cyclic quadrilateral here *(the cyclic quadrilateral ZDYC in c of **Figure 2**)*, and one cyclic quadrilateral is here *(the cyclic quadrilateral XDZA in c of **Figure 2**)*.

These intersect here *(showing point D in c of **Figure 2**)*.

PMT5: We have found this.

T4 (while working on proving)

PMT4: If we use the chords *(as saying the chords, she referred to using $\overline{XD}$, $\overline{YD}$, and $\overline{ZD}$ given in **Figure 3**)*.

…

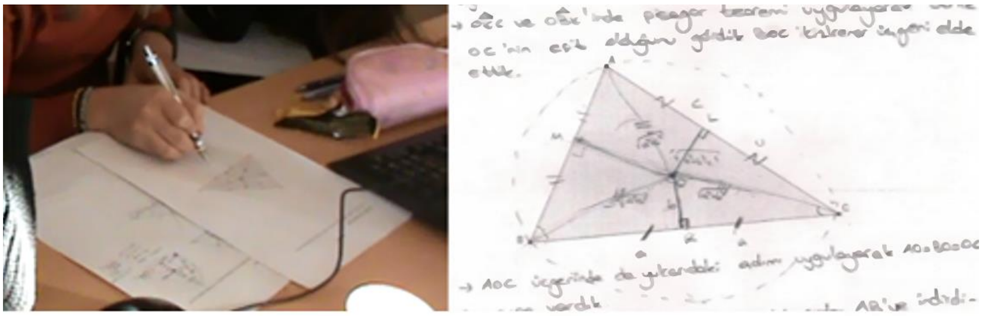
PMT6: Do you say that we draw chords like this? From this point? *(she was pointing the point D and also $\overline{XD}$, $\overline{YD}$, and $\overline{ZD}$ in **Figure 3**)*.

PMT4: We saw that this *(point D in **Figure 3**)* is the point of the cyclic quadrilaterals of two circles.

PMT5: Then, we will say that it is also for the third one *(they were searching for a way to use them in the proof)*.

The first part displayed above involves the sentences from the period that G2 was searching for the connection among the circles in T4. As seen, PMT4 and PMT5 were focusing on the GeoGebra file to search for a relationship between the mentioned circles. A screenshot from the video recordings of this period is given in a of **Figure 2** and a screenshot of the GeoGebra file of this moment is presented in b of **Figure 2**. At that point, PMT4 noticed a cyclic quadrilateral, which is composed of the points A, X, D, and Y, as in c of **Figure 2**. After G2 worked on the intended geometric figure by using GeoGebra, they explained what they had done in the given worksheet, and the geometric figure given through c of **Figure 2** is the one on that worksheet. Then, they discussed whether the sum of the opposite angles in a cyclic quadrilateral is 180°. By using the tool of GeoGebra, PMT5 calculated the sum of the mentioned angles. As the final issue related to the cyclic quadrilaterals in the geometric construction section, P6 noticed the presence of three cyclic quadrilaterals while using GeoGebra. Actually, the screenshot in b of **Figure 2** also shows three cyclic quadrilaterals as well. While explaining this to the others in the group, PMT6 drew the cyclic quadrilaterals and named each vertex, which can be seen in c of **Figure 3**. All in all, G2 noticed three cyclic quadrilaterals, which are XBYD, ZDYC, and XDZA, in c of **Figure 2**, before passing to the proving section in T4.

The second part of the excerpt given above is from the proving process of G2 in T4. As seen, while searching for a way to start the proof, P4 offered to use the chords. By saying the chords, she referred to using $\overline{XD}$, $\overline{YD}$, and $\overline{ZD}$ in **Figure 3**. **Figure 3** is given to present how G2 used the idea related to the cyclic quadrilaterals coming from the argumentation section while proving. The geometric figure was taken from the argument that G2 submitted as proof in T4. Moreover, at the end of the excerpt, G2 mentioned a way to use the cyclic quadrilaterals in proof. In more detail, they considered accepting that two cyclic quadrilaterals intersect at



(a) (b)

Figure 4. Cases from proof process of G2 in T1 (Source: (a) screenshot from video recordings of participants, (b and c) participants' own illustrations, reprinted with their permission)

point D and then deduced the presence of the third cyclic quadrilateral. By focusing on the third cyclic quadrilateral, they would bring the issue to drawing the third circle passing through point D. From **Figure 3**, it can be observed that they focused on this idea while proving.

The second example of the visual aspect is from the beginning of the proof section of G2 in T1. It instantiated the attempt of G2 to use the recently constructed geometric figures while proving.

T1 (at the beginning of proving)

PMT5: Let's open what we have done. We saved them, did not we? *(she referred to opening the GeoGebra files that they worked and saved during construction)*.

PMT4: Yes, we might use them.

As seen, PMT5 opened the GeoGebra files before starting to prove, and PMT4 agreed with her since she also saw the possibility of using them while proving. After a discussion regarding how they could prove the statement, which lasted nearly 40 minutes, PMT5 took the empty worksheet A. Then, she made the triangles in worksheet A and the GeoGebra file similar by dragging. She marked the circumcenter of the triangle printed on worksheet A more accurately by checking the location from the GeoGebra file. After that, all participants of G2 started to work on the triangle on worksheet A, as seen in a of **Figure 4**. Thus, they kept working on that worksheet and submitted the argument as proof, which was written on a triangle printed on the worksheet, as presented in b of **Figure 4**. It can be inferred that G2 preferred to use the triangle that they were familiar with via geometric construction while proving the related statement in the activity.

The last code of the affordances is the veracity of the statement. In more detail, since participants were sure that the statement in worksheet B of the tasks was true, they did not consider finding a counterexample. To provide an example, the excerpts from both argumentation and proof processes of G1 in T3 are presented below.

T3 (at the end of conjecture production)

PMT1: I did not know that they *(the orthocenter, the circumcenter, and the centroid of a triangle)* are collinear.

PMT2: Me too.

PMT3: I did not know too.

T3 (at the beginning of proving)

PMT2: Here, can we do this? By accepting that two of them *(two of the points)* are collinear, can we say that if this and this are collinear, then this one is also collinear? *(she considered drawing a line from two points and then tried to show that the third one was also on that line)*.

…

PMT1: I am thinking about whether I can arrange any congruence among three lines.

PMT2: Three points.

PMT1: Does the congruence work again for this one? *(since they arranged congruence in the previous tasks, she also searched for any congruence for this proof)*.

…

PMT1: If I accept that these are not collinear, then it turns out that they are collinear. However, when I accept that they are not collinear, how can I show that they are collinear? *(she referred to using proof by contradiction)*.

The first part of the excerpt indicated that they did not know that the orthocenter, the centroid, and the circumcenter of a triangle are collinear. All participants of G1 learned it at the end of the conjecture production section of the task. Thus, at this point, G1 produced the conjecture related to the collinearity of the mentioned three points so that they were sure about the veracity of the statement due to being involved in such an argumentation process.

The second part of the excerpt given above is from the beginning of the proving process of G1 in T3. It covers three ideas to start to prove. As the first idea, PMT2 stated whether they could try to draw a line from two points and then show that the third one was also on that line. Secondly, PMT1 offered to look for a congruence that might be used in the development of proof. In the last idea, PMT1 referred to using proof by contradiction without stating the name literally. That is, she mentioned accepting that the orthocenter, the centroid, and the circumcenter of a triangle are not collinear and trying to reach the collinearity of them. All ideas mentioned for proof aim to prove that the statement is true. It can be stated that they did not consider showing that the statement is false since they did not present any evidence for the presence of a counterexample or the need to find a counterexample. Since they produced the statement by means of geometric construction embedded in the argumentation process, they were sure about the veracity of the statement asked to prove. The issues originating from the argumentation might affect the method of proof they focused.

Constraints of Argumentation Process before Formal Proof Process

Similar to the affordances explained, the first code of constraints is negative affective occasions. Particularly, participants were sometimes bored during the first section of the tasks. It reflected on the proof process, they presented evidence of frustration when they got stuck, and they constituted the negative attitude toward the whole task. It might be stated that the mentioned cases affected the proving sections of the tasks in a negative manner.

As an example of this code, practice of G1 in T4 is shown. It was seen that many negative sentences of G1 were listed in code.

T4 (during conjecturing)

PMT2: I could not understand what kind of a relationship we might find.

…

PMT2: I have always found the same thing at the end, then my psychological status is failing.

...

PMT2: I think, I could not draw it. Yes, I could not. Here we go, I am drawing it wrong again.

…

Instructor: You are so calm today. Do you have a problem?

PMT1: We could not produce anything today.

Instructor: What do you think about the relations of circles?

PMT1: I wish, we could think of something related to them.

T4 (while working on proving)

PMT2: How am I supposed to prove this?

…

PMT3: You will complete it to 180. This one is α, then plus β, it becomes 360. However, I could not prove it.

The first issue in the excerpt is related to the negative attitude of PMT2 while searching for a relationship among the circles. She declared that she neither understood what she was supposed to find nor drew the intended figure correctly. Since she kept finding the same result at the end of different attempts, she directly stated that she was affected by this situation negatively. Since the instructor noticed the overall negative status of G1, she asked whether they had a problem. As the answer, PMT1 stated that they could not produce today. Then, by aiming to enhance the argumentation, the instructor asked for their ideas related to the connections of circles. PMT1 gave another negative sentence since she signified that they could not find it yet. Although they were able to produce the aimed conjectures by performing the constructions correctly, their status of being bored during the conjecture production process was maintained in the proving section of T4. As seen, the second part of the excerpt is from the proving process of G1. As soon as the worksheets were distributed, PMT2 complained about the proof. That is, she declared “how am I supposed to prove this” with a frustrated intonation. Similarly, in the followings of the proving process, PMT3 stated that she could not prove it a few times.

The second example is from both the argumentation and proving process of G2 in T1. It covers some negative sentences of PMT5 and PMT6.

T1 (during conjecturing)

PMT5: Dash it, it did not work *(an approach she tried to perform the intended geometric construction did not work)*.

…

PMT5: We drew this. Is this might be related to the one inside? *(she was talking about the approach focused)*. I could not think.

PMT6: I did not have any idea too. I get stuck. I noticed that I am not good at this concept of geometry. I do not know much about it.

T1 (at the beginning of proving)

PMT6: We have barely found it *(the statement asked to prove)*. How can we prove it now?

In the first part of the extract, PMT5 gets frustrated since a method she believed in much did not result in a geometric construction. After that, they kept looking for an approach to perform the geometric figure asked in T1, which is the circumcircle of a given triangle. However, there were moments when they got stuck and presented low self-efficacy in terms of conducting the whole activity. PMT6 declared that she was not good at the concept of the activity. Although they found a working approach and concluded that the intersection of the perpendicular bisectors of the sides of a triangle is the circumcenter, which was the aimed conjecture in the activity, they still had a negative stance towards the activity. As seen in the second part of the extract given above, PMT6 said out loud her concern related to their capability of conducting the proof of the given statement.

Lastly, the confusion related to the difference between conjecturing and proving was listed as a code among the constraints of the argumentation process before the proving process. According to the analysis, this code was rarely seen in the data collected. To provide an example, the extracts from the proving process of G1 in T1 are given below.

T1 (at the beginning of proving)

PMT2: What do I prove here? Is not it already obvious?

…

Instructor: You will present formal proof for the statement you have constructed in the last section.

PMT3: We have to produce the congruent angles, triangles, etc. here. What can we do?

T1 (while working on proving)

PMT2: What if we transfer the angles? How do we transfer the angles?

PMT1: You do not have to conduct something by using compass while proving.

…

PMT2: Okay, I think that we perform geometric construction during proving. I sometimes get confused.

The first part of the excerpt is from the beginning of the proving process of G1 in T1. As indicated in the first sentence, PMT2 could not see the difference between conjecturing and proving. Since G1 produced the statement by virtue of geometric construction in T1, she could not see the need for proving. That is, according to her, they had already presented that statement. After this idea of PMT2, both the instructor and PMT1 tried to show the point that she missed. After that, in the following moments of the proving process, PMT2 presented another confusing point for her, which is the second part of the extract. She attempted to transfer the angle while trying to prove it. Again, PMT1 warned her by stating that there was no need to use a compass while proving. Thus, PMT2 noticed what she was confused about.

DISCUSSION

As stated, this study aims to investigate how prospective mathematics teachers' argumentation while producing conjectures relates to their formal proof process in the context of geometry. For that purpose, in the data analysis, the affordances and constraints of being involved in argumentation before formal proof were portrayed. In this section, four codes in the affordances and two codes in the constraints are discussed.

In this study, some affective occasions were observed as both an affordance and a constraint of being involved in an argumentation process before engaging in the formal proof process. In geometry-proof tasks, it is the flow of the argumentation of groups that specifies the conjectures. Accordingly, the aforementioned argumentation process determines what would be proved in the following proving section of the tasks. This might be the reason underlying the participants' confidence in providing proof of the statements. That is, they might think that they were able to find the statements that were asked to be proven by exploring and performing constructions; hence, they were also capable of conducting the proofs of these statements. On the other hand, in cases, where the conjecture production process was considerably difficult for them and they got bored during the process, they might have been reluctant to work on the proving phase even though they had found the statements. As mentioned, the participants worked as a group. If a productive, supportive, and motivational atmosphere can emerge among the participants during the collective argumentation process, this might increase their motivation in the proving section of the tasks. However, it has been generally reported that proof is a difficult concept for students at all levels (Ellis et al., 2012). In this respect, PMTs' affective stance during the argumentation process might be considered critical since proof is already a difficult and tedious task.

As a code in affordances, it was concluded that the participants had the opportunity to arrange the knowledge base related to the concepts of the tasks by being involved in the argumentation in which conjectures were produced. While looking for the properties of geometric figures, the participants could check whether their previous knowledge or presuppositions were correct. Developing students' reasoning and thinking skills and supporting them in developing a predisposition towards producing conjectures and proposing related plausible arguments are critical because these issues constitute the basis for further experiences (NCTM, 2000). Therefore, conjecturing can be considered a point of entry into both an activity and a reasoning process in the mathematical domain. Thus, to participate in an activity that involves the production of conjectures might offer the opportunity to activate their hidden cognitive processes.

The visual aspect is presented as the third code in affordances (see **Table 2**). Garuti et al. (1996) proposed the presence of a possible cognitive continuum between the production and proving processes of a statement. Within the context of geometry, visualization in the argumentation process might be considered a critical issue in terms of the mentioned continuum. Sinclair et al. (2012) stated that "geometric images provide the content in relation to which properties can be noticed, definitions can be made, and invariances can be discerned" (p. 8). In this respect, in the argumentation process, the participants of the study might notice many cases visually, work on extra and even unrelated cases of the concepts before the proving section and see the cases, which have the potential to help them while proving. It was highlighted that the development of students' geometric reasoning and awareness is directly related to construction and visualization (Sinclair et al., 2012). Thus, while working on the geometric constructions in the present study, the participants might have developed their geometric reasoning with the help of visual elements before the proving process. Another point related to the visual aspect is the auxiliary lines, which Fan et al. (2017) mentioned. They underlined that "in geometric proof, adding auxiliary lines is often helpful and in many cases necessary" (p. 230). In the present study, PMTs might have had the opportunity to see the potential auxiliary drawings, which may have been used while proving.

Last point to mention in the affordances of experiencing argumentation before the formal proof is related to the veracity of the statement. The examination of the veracity of the conjectures is one of the main steps of the conjecture production process. It can be stated that PMTs in the present study were sure about the fact that the statements given to them for proof, which were also produced by them recently, were true. Thus, they did not focus on the idea of finding counterexamples. A consistent finding was reported in a study of Boero et al. (1995). They observed that middle grade students who participated in a conjecture production process started to work on the proving stage one step ahead since they already had the idea related to the validity of the statement they aimed to prove.

Since affective occasions were discussed as a whole, confusion related to the difference between conjecturing and proving is the last code in the constraints to discuss herein. As mentioned, the participants were rarely seen to experience difficulty in differentiating the concepts of conjecturing and proving. This result coincides with the finding reported in a previous study of Pedemonte (2008), indicating that prospective primary school teachers might be experiencing difficulty in proof due to the fact that they cannot transform the argumentation structure used during conjecturing into a deductive structure. In a similar vein, the participants of the present study might have tried to transfer their actions during conjecture production to the proving phase as well. Moreover, this result might have resulted from the fact that they continued to think in a way that they were still endeavoring on the geometric constructions in the proving section of the task.

Based on the findings of the study, the contents of the courses in mathematics teacher education programs that involve argumentation and proof might be revised. Similarly, this study could contribute to the development of the contents of geometry related courses in mathematics teacher education programs. In more detail, the content of such courses could be modified according to the characteristics of prospective mathematics teachers' argumentation and proof processes and the relations between these processes. Moreover, when the issue is not advanced mathematics, but middle school students, both negative and positive aspects of the relations between argumentation and proof could be taken into consideration by mathematics teachers to improve students' reasoning and design activities that end up with informal proofs or justifications.

Limitations and Future Directions

Some issues could be regarded as the limitations of the study are explained as follows. The first issue, which can be considered as the limitation is that the data was collected from six junior prospective mathematics teachers in a state university in Ankara. The findings of the study would be different if some other juniors were selected from either the subject university or another university. In a similar vein, if the participants were selected from other year levels in the same program such as freshmen, sophomores, and seniors, the findings would be different. Moreover, since the participants were selected by purposeful sampling, they might not be representative of other prospective mathematics teachers. In this respect, the findings can be considered less generalizable to other occasions. Actually, it can be stated that the generalization was not the aim of the study due to its qualitative nature. Another limitation of the study is that the data were mainly gathered by means of geometry-proof tasks. Since data collection was centered on these tasks, which were prepared as related to triangles and circles, the findings of the study are limited to these geometry concepts. By considering the findings, implications, and limitations of the study, some recommendations for further research are presented below.

In this study, how argumentation relates to proof was considered within the geometry domain. Therefore, in further studies, it might be worth investigating this issue with the participation of different levels of students and within the other areas of mathematics. Indeed, drawing on the findings of the studies conducted by Pedemonte (2007, 2008), which pointed out some differences between geometry and algebra in terms of cognitive unity, such studies conducted with other mathematical domains have the potential to contribute to the related literature and provide comparisons. Thus, it was highly recommended that the relations between argumentation and proof should be considered within the scope of the other domains of mathematics. As an extension of this study, it might be investigated how prospective mathematics teachers prepare activities or lesson plans on triangle and circle related objectives in middle school mathematics curricula by aiming to integrate the conjecturing and dynamic exploration like the ones they have been involved in the geometry-proof tasks. Since the middle school mathematics curriculum does not cover formal proving, prospective mathematics teachers might focus on producing conjectures, reasoning, and justification. Such a study might also fill the gaps in terms of practical considerations.

CONCLUSIONS

According to the codes that emerged in the findings, it can be stated that there are both affordances and constraints of being involved in the argumentation process before providing proof in the context of geometry. It was also seen that the themes reached in the data analysis did not differ in terms of the groups. Since how the groups' argumentation process while producing conjectures relates to the proving process of conjectures is not an issue directly dependent on the tools used during geometric constructions, similar codes might be seen in both groups during the mentioned analysis. It might be better to consider that how argumentation relates to proof is an issue dependent on the mathematical domain, which is geometry in the present study

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