International Electronic Journal of

Mathematics Education

Volume 4, Number 3, October 2009 www.iejme.com

Special issue on "Research and Developments in Probability Education"

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CHANCE ENCOUNTERS – 20 YEARS LATER
FUNDAMENTAL IDEAS IN TEACHING PROBABILITY AT SCHOOL LEVEL

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ABSTRACT. This paper considers how probability is now taught in England and the way that the curriculum reflects key research ideas from the last few decades. Links are made to work undertaken in probability education and the way that challenges in the book, Chance Encounters, have been met. This is based on the current curriculum and also the performance of children in tests. The key question considered is the extent to which the teaching of probability has changed over the last twenty years. The conclusion notes that there is some way to go in ensuring children are well versed in probability.

KEYWORDS. Probability, secondary school, teaching.

1. INTRODUCTION

It is almost two decades since the research undertaken for Chance Encounters (Kapadia & Borovcnik, 1991); the title was chosen as slightly whimsical but the underlying content of the teaching of probability was promoted and assessed critically and carefully. This was clear from the sub-title of Probability in Education. The book was written to collect and share findings from major researchers across Europe and North America on key aspects to the teaching of probability internationally. It grew out of collaborative work at the first two International Conferences on the Teaching of Statistics (ICOTS) in the 1980s, in Sheffield and Vancouver respectively, supported

ISSN: 1306-3030

by smaller meetings elsewhere. A deliberate attempt was to take perspectives from various viewpoints, not least from the origins of the subject with regards to games, gambling, and counter-intuitive paradoxes. Links were made to the current curricula at the time, including the National Curriculum of Teachers of Mathematics (NCTM) in the United States. Attention was paid to key psychological research and to the growing influence of computers.

In the book a number of challenges were set. This paper aims to review those challenges, with specific commentary on the situation in England. This is not intended as an exhaustive or comprehensive study but rather to make some comments to provoke and stimulate debate. Many papers, books and articles have been published but no attempt is made at a summary or detailed literature review, though that would be an important task to complete. Early research in concepts in probability was by Kapadia & Hawkins (1982), or Green (1982), and partially summarised by Fischbein et al (1997).

Perhaps the key point to make is the changing focus in England at least. The focus in England has changed towards a greater emphasis on statistics and statistical methods, particularly the representation of data, partly at the expense of probability. Probability is taught in a relatively limited way and not systematically linked as the underpinning theory for statistical distributions. The reasons for this change are not entirely clear. One reason may be that probability, as taught by mathematics teachers continued to be seen as rather abstract and theoretical. This links to the second reason that statistical data pervades society increasingly and so is seen as more relevant to teach.

However, across the world risk and management of risk is being seen as increasingly important. The research of Tversky & Kahneman (1974) is now recognised as ground-breaking, not least with the award of the Nobel prize, following the substantial research in their book – Kahneman, Slovic, & Tversky (1982). Its importance and relevance in economics is growing. Yet education has not fully incorporated the underlying ideas.

There were ten provocative statements relating to how people view, use and process ideas relating to probability (see below). What has changed? What is actually taught and what can children do currently? The responses to these questions will relate to current policy in England. The aim is to stimulate responses from those in other countries about their own perspectives by subsequent written communication and research. This paper gives some thoughts but, more importantly, it aims to promote a more sustained debate about the teaching and learning of probability at secondary school level (grades 6-10).

2. CURRENT CURRICULUM: ENGLAND

Below (in Appendix 1) are the attainment targets and related programmes of study in England for probability. Attainment targets are similar to the term standards as used in USA and other countries: attainment targets are expectations to be reached by pupils at certain ages or stages of their education. The attainment targets cover 8 levels (1-8), corresponding to the primary and secondary stages respectively. There are four levels for each stage but the expectation is that most pupils would cover four levels at primary school and two or three at secondary school.

Level 4 is the expectation for pupils at the end of primary school in Year 6 (grade 5) or age 11. Level 5 is for Year 9 (grade 8) or age 14; Level 7 is for Year 11 (grade 10) or age 16. In fact, in England, a range of levels is expected at each age, since pupils almost invariably move from one year to the next, irrespective of performance on internal or external examinations and tests. This is different in some countries where pupils have to reach a certain level at the end of each year before moving to the next year. Hence, Level 8 and exceptional performance is for pupils with high ability, above the top 15th percentile. Most pupils would not be expected to reach this level at the end of secondary school.

The attainment targets describe what pupils are expected to learn and the content covered in tests. The more detailed programmes of study include more general information on what should be studied. There is also information (in Appendix 2) about performance in a few questions on probability in tests undertaken at age 14 in 2001.

A key point to note in terms of children up to Year 6 (working at Level 4) is that there is no reference in the attainment targets to probability, even at a basic level. This is a marked change from the curriculum as originally devised twenty years ago, when even infants in Year 2 were expected to investigate some early notions of chance. At that time the assessment criteria were very detailed and specific, with a whole set of attainment targets which related solely to probability; at the time it was attainment target 14, whilst attainment targets 12 and 13 related to statistics. Now there is a single set of targets devoted to handling data which is seen to encompass probability. In some ways the merger could be seen in a positive light, except that some key ideas of probability (such as subjective notions) were removed. Most importantly, because of external pressures on an already crowded curriculum, direct references to probability were initially removed from the content for infants and then for all children up to Year 6 (age 11), with regards to attainment targets. These changes were implemented early this century and so it would be interesting to conduct tests on children at different stages of school to monitor any effects.

Probability has been retained for secondary age pupils as can be seen in the attainment targets for Level 5. Work would begin in Year 7 and continue for the next year on the ideas of probability derived from equal likelihood and experimental situations only. While this is a standard way of addressing probabilistic notions, it is unfortunate that no direct mention is made of subjective notions, particularly in situations where an experiment is not possible. Moreover, it is also important to confront children's early conceptions and misconceptions of probability. This would normally be taught in schools (as illustrated in the programmes of study), but explicit mention of subjective probability would have been particularly helpful. Another consequence of this shift to later school years lies in that playful activities accompanying learning might be more suitable for younger children and not attract the interest of older children in secondary schools.

The next level (Level 6 for Year 9) for probability develops into combined events and notions of mutual exclusiveness. It is only at Level 7 for Year 11 that mention is made of distributions and inference, though links with probability are not explicitly mentioned. There is a substantial gap between the aforementioned "combined events" (at Level 6) and "distributions and inference" (at Level 7); equally it is not clear how distributions are approached if links with probability are not explicitly mentioned and securely taught previously. Moreover, it is only for the able pupils that ideas of independence are explicitly studied, as well as aspects of sampling, with links to reliability. In practice, sampling would be introduced earlier for surveys, but the key links between probability and statistics may not be made explicitly.

Hence the content of the current curriculum is relatively brief. It does give an overview but lacks important detail. Perhaps the most important omission is any reference to risk and hence to real-life applications. It is certainly true that the majority of teachers would make links to real-life situations, indeed they often occur in text-books. But the lack of reference to risk is very important in the context of the current world.

It is for this reason that the research of Tversky & Kahneman has come to such prominence. It is recognised as a key skill in many situations; too often it is a key skill which is lacking, even in students who study for higher education and will therefore go on to the key posts of influence and responsibility. If risks are attached to undisputed probabilities interpreted as frequencies, they may not be so problematic. The more subtle issue is when risks are attached to subjective probability, an element which has disappeared as an explicit element of the curriculum.

Though there are no specific attainment targets in probability for Year 6 pupils, ideas of equal likelihood are included in the programmes of study described below. However ideas of probability are not included in tests for Year 6. Since the tests for Year 6 pupils are perceived as 'high-stakes' tests which are used to gauge the performance of a school, it may be that younger children do not now get the exposure to underlying ideas of probability sufficiently.

It is also crucial to note that there is virtually no reference to subjective probability at any stage. This omission is very significant for various reasons. Subjective notions of probability are the first notions that many children develop, albeit on an informal basis. There are two aspects of subjective notions; one relates to intuitive conceptions; the other is that in many applications involving risks, probability is neither objective nor open to a frequency interpretation.

Indeed we all use informal probabilistic notions daily in making certain decisions. Thus it is important when studying ideas of probability in the classroom, some account should be taken of such perceptions. This is a fundamental way of tackling misconceptions. One common and immediate such misconception relates to the chance of getting a six when throwing a die. This is linked to waiting times for a desirable event; the waiting seems to increase the internal perception of likely occurrences.

Waiting time, however, is but one other – objective – feature of the situation, which is more important to children as they actually have to wait for long times occasionally. For example, the probability of waiting for 6 throws before a six comes is about 1/3, which is not infrequent. Moreover, one would remember such an occurrence much more readily than the (more likely) number of occasions when one gets a six within three throws. This does not refer to subjective probabilities nor to the subjective re-construction of the situation. It links to the 'availability' heuristic, as identified by Tversky & Kahneman. It is not easy to appreciate such high waiting times with $p = 1/6$. The calculation of such probabilities is technically a time-consuming and rather complex task.

In all research in (mathematical) education, it is recognised that early conceptions of ideas need to be discussed so that children can develop better understanding. Probability is no different and early notions as well as misconceptions need to be addressed. This takes time which is not fully recognised. This approach would also clarify the aims, purpose, and limitations of probability calculations. It might also increase the rate of acceptance of such awkward calculations as it becomes clearer what they really stand for.

The approach taken in these attainment targets and programmes of study is still quite firmly based on relatively formal and mathematical notions of probability, where contexts and underlying subjective knowledge is to be ignored. In some ways, this is linked to the fact that it is easier to set questions in such contexts. The questions are seen as reliable and consistent in testing concepts and ideas accurately. However, it is a moot point as to how well children can use such ideas in the future for real-life situations. This becomes clear when looking at the performance on the test questions and children's responses given below.

3. TEST RESULTS

In the Appendix 2 below are quotations from analysis of test results relating to probability of Year 9 (age 14 who are expected to be at Level 5 or 6) pupils in 2001; as noted above, they would have encountered ideas of probability in their earlier primary education. It will be interesting to compare the performance on similar ideas of those who take the tests from next year, as they may not have encountered ideas of probability in earlier primary school education. This synoptic comparison with 20 years before could be re-visited after a few years as the conceptual critique of curricula would then be accompanied by an empirical analysis of its impact. This later study would allow for new conjectures to be explored. It may be that pupils are less able to reconcile subjective and frequentist notions of probability.

In particular the analysis of facility on questions does show good intuitive understanding of the basic underling notions of probability, particularly in experimental situations. The difficulties seem to arise in relation to pupils' facility and knowledge of other ideas in arithmetic and geometry. There is also evidence of the use of adding rather than multiplying given or derived probabilities even when it may lead to wrong answers such as a probability exceeding unity.

For the first set of questions, equal likelihood in simple situations is effectively used, with the correct notation by the vast majority (around three quarters) who reach Level 5 in Year 9. The low facility for the third question, *Coloured Cubes*, may well be related to poor facility with fractions rather than notions of probability. *Coloured Cubes* was targeted at level 6 and in part (c) given an unspecified number of cubes that are either black or red and the probability that one taken at random being red is 1/5, pupils at level 6 found some difficulty understanding that the same probability could arise from different total numbers of cubes. In part (d) pupils were told that there are 20 cubes including three different colours, and with a given probability of obtaining green; they were asked to estimate the maximum number of cubes of a second colour.

The solution to this task involves two markedly different steps, first to solve for the number of greens, then to solve for the maximum number of yellows.

About half the pupils achieving Level 6 could perform the first step, but only a third could complete the second step.

The more discriminating and interesting test for equal likelihood occurs in Appendix 2 (*Spinners*). For an interpretation of test results, see also QCA key stage 3 (pp. 17). In part (a) of the questions, pupils were shown two spinners. The first, a regular pentagon, was divided into five similar triangles numbered 1 to 5. The second, a regular hexagon, was divided into six similar triangles numbered 1 to 6.

In part (b) pupils were shown two similar hexagonal spinners although one was larger than the other. The errors children make are linked to using the notion of area rather than angle in the question. In fact area is often the right measure to use in determining equal likelihood. In this particular case, one spinner is made bigger than the other one, almost deliberately to trick children. Whilst there are times when a trick is sensible to use to search for underlying understanding, this is not apparent here. The key concept is to either choose one of five or one of six equally likely outcomes. For such a situation, to look for underlying probabilistic understanding, it would be better to have spinners of similar size; this would make the item less discriminating with regards to children's facility with the question, but more closely linked to the underlying ideas.

To have the different size of the spinners does indicate that children still have their irrelevant thought about random devices and experiments. That this has such a big influence is a sign that teaching has not been fully effective. Their reasoning is still blurred with a lot of irrelevant ideas, the size of spinners is but one of them.

For this question, it is also interesting that children are asked to give a reason to explain their answer. This is particularly powerful to help to understand a child's reasoning and used too rarely. In this case, the notion of the starting point of the arrow is a misconception which needs to be tackled from a probabilistic perspective, particularly if it accounts for a significant proportion of children.

The question on *Pots* is a typical one about combining probabilities, perhaps by using a tree diagram; see also Appendix 2. At level 8, the question entitled *Pots* assessed the probability of combined events. Pupils were given the probability 0.03 of a pot cracking when fired. In part (a) they were asked to calculate the probability of both pots cracking when two were fired. Part (b) asked for the probability of only one pot cracking when two were fired. This question is aimed at the high ability pupils, typically the top half of the population. (See QCA key stage 3, pp. 17.)

The first part is an interesting and sensible test of deciding when two probabilities should be added or multiplied in a given situation. It is a key idea in moving from the probability of a single event using equal likelihood or frequentist notions of probability. Since two fifths of pupils incorrectly added, this implies that almost three quarters (a very high proportion) of Year 9 children have incorrect notions of the fairly fundamental notions of when it is inappropriate to add two probabilities. The final part of the question is certainly quite difficult and challenging. The facility for the full answer is very low and represents about the top decile (or less) of the population, whilst around the top 5% got the question only partially right in forgetting that either the first or the second pot could crack.

For each of the above questions, it is also useful to undertake a more detailed complexity analysis (see appendix) of the questions. In a test situation, there have to be answers which are designated as either correct or wrong. This is part of the nature of testing. However, a different analysis is required in trying to understand the light shed on pupils' understanding from the answers they give. Indeed, some of the answers marked as wrong may simply reflect a different interpretation and may even be seen as correct in certain scenarios.

One example of such an anomaly is given above with the presentation of two spinners of different sizes, leading children to look at absolute rather than relative areas rather than the angles. In this case children were also asked to give a reason for their answer. Unfortunately, limited details are provided on the answers given by children. This is a pity. A false answer or a correct answer does not always tell much. The key question is more often what reconstruction the pupil used in order to derive a given answer; related to that, the answer could well be correct. Or, correct answers could be given by just using the "right" operation without understanding. Similarly, and more controversially, some questions make an implicit assumption about independence of events which may not be justified in reality.

With regards to the challenges set out in Chance Encounters, the curriculum as set out in its assessment objectives, does not show signs of addressing key areas of misconceptions, which was a vital strand running through the book. This indicates that insufficient notice has been taken of the psychological issues explored, or the teaching sequences discussed. Another notable omission relates to computers, though such references would, more properly, come in the supporting curriculum materials. The book also contained the following ten provocative statements at the end: most remain all too true. Elements of being haphazard and incomplete in dealing with probability are clear and people use memorable events to process information. Very low and high probabilities are hard to assess. Frequentist ideas underlie much of the approach to probability and people are getting better in using such information.

4. TEN STATEMENTS

Chance Encounters (Kapadia & Borovcnik 1991) ended with the following set of ten provocative statements. It is clear from current research that many of these issues still need to be addressed in school.

1. People use personal experience in assessing chance in a rather haphazard manner
2. People process information in a rather incomplete way
3. People process information in a way biased by memorable events
4. People find it hard to assess probabilities which are very low or very high
5. People do not assign values of 0 for impossibility and 1 for certainty
6. People equate certainty and impossibility with physical rather than logical events
7. People equate 50-50 chances with coin tossing
8. People assign equal likelihood in unknown situations
9. People are incoherent in assigning and in processing probabilities
10. People are supra-additive

- The current curriculum in England does begin to address the first two statements in the secondary school stage, but moves too rapidly on to more formal calculations; this is one reason for beginning to explore such ideas in the primary school.
- There are few concrete approaches to the third statement and there are many real-life examples which could be explored. It is hard to know of the best way to explore the fourth statement; there is a need for research in this area, perhaps related to notions of risk.
- The fifth statement has been addressed well.
- The sixth one is perhaps only relevant for those pupils who go on to study mathematics at a higher level.
- The seventh remains true but can only be addressed by experiential approaches. It can be that the approach in the eighth statement is all that is possible.
- The last two statements about manipulation of probability require careful and sustained practice, built on firm conceptual foundations of the meaning of probability; however, this requires careful and deep discussion in the classroom, a relatively rare occurrence in mathematics lessons.

For actual teaching it would be informative why people behave as in the list indicated. This would foster better endeavours in teaching.

5. CONCLUSION

In England, at least, there were positive steps taken towards the teaching of probability twenty years ago, starting with young children learning the intuitive underlying ideas. This has changed and is now mainly for secondary age pupils, where a more formal approach is taken too quickly. There is insufficient attention given to the fundamental ideas in probability and in addressing misconceptions. The test results show that pupils do gain facility in situations where equal likelihood can be assumed. The errors made indicate that subsequent ideas of combining probabilities are challenging to most pupils. The errors also show that misconceptions on key ideas remain.

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APPENDIX 1: NATIONAL CURRICULUM: ATTAINMENT TARGETS 2001
MATHEMATICS KEY STAGE 3

Handling Data and Probability

Level 4 (Year 6, grade 5, age 11)

Pupils collect discrete data and record them using a frequency table.
They understand and use the mode and range to describe sets of data.
They group data in equal class intervals where appropriate, represent collected data in frequency diagrams and interpret such diagrams.
They construct and interpret simple line graphs.

Level 5 (Year 9, lower ability, grade 8, age 14)

Pupils understand and use the mean of discrete data.
They compare two simple distributions using the range and one of mode, median or mean.
They interpret graphs and diagrams, including pie charts, and draw conclusions.
They understand and use the probability scale from 0 to 1.
They find and justify probabilities and approximations to these by selecting and using methods based on equally likely outcomes and experimental evidence, as appropriate.
They understand that different outcomes may result from repeating an experiment.

Level 6 (Year 9, average ability, grade 8, age 14)

Pupils collect and record continuous data, choosing appropriate equal class intervals over a sensible range to create frequency tables.
They construct and interpret frequency diagrams.
They construct pie charts.
They draw conclusions from scatter diagrams, and have a basic understanding of correlation.
When dealing with a combination of two experiments, they identify all the outcomes.
When solving problems, they use their knowledge that the total probability of all the mutually exclusive outcomes of an experiment is 1.

Level 7 (Year 9, high ability, grade 8, age 14)

Pupils specify hypotheses and test them by designing and using appropriate methods that take account of variability or bias.
They determine the modal class and estimate the mean, median and range of sets of grouped data, selecting the statistic most appropriate to their line of enquiry.
They use measures of average and range, with associated frequency polygons, as appropriate, to compare distributions and make inferences.
They understand relative frequency as an estimate of probability and use this to compare outcomes of experiments.

Level 8 (Exceptional performance, Year 11, grade 10, high ability)

Pupils interpret and construct histograms.
They understand how different methods of sampling and different sample sizes may affect the reliability of conclusions drawn.
They select and justify a sample and method to investigate a population.
They recognise when and how to work with probabilities associated with independent, mutually exclusive events.

Programmes of Study

Handling data: Year 6

- Use the language associated with probability to discuss events,
  including those with equally likely outcomes.
- Solve a problem by representing, extracting and interpreting data in tables, graphs, charts and diagrams, including those generated by a computer, for example:
  – line graphs (e.g. for distance–time, for a multiplication table, a conversion graph, a graph of pairs of numbers adding to 8);
  – frequency tables and bar charts with grouped discrete data (e.g. test marks 0–5, 6–10, 11–15…).
- Find the mode and range of a set of data. Begin to find the median and mean of a set of data.

Probability: Year 7

- Use vocabulary and ideas of probability, drawing on experience.
- Understand and use the probability scale from 0 to 1;
  find and justify probabilities based on equally likely outcomes in simple contexts;
  identify all the possible mutually exclusive outcomes of a singe event.
- Collect data from a simple experiment and record in a frequency table;
  estimate probabilities based on this data.
- Compare experimental and theoretical probabilities in simple contexts.

Probability: Year 8

- Use the vocabulary of probability when interpreting the results of an experiment;
  appreciate that random processes are unpredictable.
- Know that if the probability of an event occurring is p, then the probability of it not occurring is $1-p$;
  find and record all possible mutually exclusive outcomes for single events and two successive events in a systematic way, using diagrams and tables.
- Estimate probabilities from experimental data; understand that:
  – if an experiment is repeated there may be, and usually will be, different outcomes;
  – increasing the number of times an experiment is repeated generally leads to better estimates of probability.
- Compare experimental and theoretical probabilities in different contexts.

Probability: Year 9

- Use the vocabulary of probability in interpreting results involving uncertainty and prediction.
- Identify all the mutually exclusive outcomes of an experiment;
  know that the sum of probabilities of all mutually exclusive outcomes is 1 and use this when solving problems.
- Estimate probabilities from experimental data; understand relative frequency as an estimate of probability and use this to compare outcomes of experiments.
- Compare experimental and theoretical probabilities in a range of contexts;
  appreciate the difference between mathematical explanation and experimental evidence.

Probability: Year 9 Exceptional Performance

- Understand relative frequency as an estimate of probability and use this to compare outcomes of experiments.

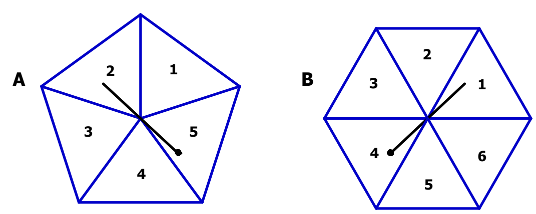
APPENDIX 2: PROBABILITY TESTS FOR 14 YEAR OLDS 2001

(QCA 2001; search the site by subject "mathematics".)

1 The performance of pupils at level 4 on *Tokens*, where they considered probabilities relating to drawing gold or silver tokens from a bag, was good and showed a sound understanding of this context and probabilities expressed in words. Pupils who achieved level 5 and above could confidently find the probability of an event not happening, given the probability of its happening, though this concept is at level 6 of the national curriculum. Another question called *Cereal* was targeted at level 5 and concerned writing probabilities relating to four equally likely events. This question was found straightforward for pupils at level 5, which is reached by about 75% of the cohort. It was pleasing that few pupils lost marks through incorrect notation for probability.

2 In part (a) of the question *Spinners*, pupils were shown two spinners. The first, a regular pentagon, was divided into five similar triangles numbered 1 to 5. The second, a regular hexagon, was divided into six similar triangles numbered 1 to 6. In part (b) pupils were shown two similar hexagonal spinners although one was larger than the other. At a lower level, pupils' explanations to parts (a) and (b) revealed some common misconceptions among pupils at level 4. One of the questions is given below:

"Spinners (a) The diagram shows spinner A and spinner B.



Which spinner gives you the best chance to get 1?

Tick your answer.

spinner A   spinner B   doesn't matter

Explain why you chose that answer." (See QCA, pp. 17)

In part (a), a common incorrect response was an indication that it did not matter which spinner you used to have the best chance of getting 1. This suggests, perhaps, that these pupils had seen the sections labelled 1 as having the same angle at the centre (perhaps showing a lack of attention to detail). A common incorrect reason for choosing the correct spinner A was that the arrow on A was closer to 1 (showing that some pupils use redundant information incorrectly).

In part (b), the common error was to indicate the larger spinner. This error was made by almost 40% of pupils at level 4 and over 20% of those at level 6 and it suggests that pupils have compared the areas of the sections labelled 1 on both spinners, rather than the angles at the centres of the sections. Overall, the errors can be linked to specific aspects: use of irrelevant facts, incorrect knowledge of fractions or percentages and using an incorrect visual stimulus.

3 *Coloured Cubes* was also targeted at level 6 and in part (c) given an unspecified number of cubes that are either black or red and the probability that one taken at random being red is 1/5, pupils at level 6 found some difficulty understanding that the same probability arises from different total numbers of cubes. This shows that the ideas are quite complex. Part (d) of these problems is given below:

> A different bag has blue (B), green (G) and yellow (Y) cubes in it. There is at least one of each of the three colours.
> The teacher says:
> 'If you take a cube at random out of the bag, the probability that it will be green is 3/5.'
> There are 20 cubes in the bag.
> What is the greatest number of yellow cubes there could be in the bag?
> Show your working."

In part (d) pupils were given the context of 20 cubes with at least one of each of three different colours (green, yellow and blue). Given the probability of drawing a cube of one colour at random they were asked to find the greatest possible number of a second colour. They were told that the probability of a green counter was 3/5 and asked to find the maximum number of yellow counters (7).

Overall only one-eight of the cohort was awarded both marks for part (d). This shows that pupils experience difficulties when a problem has several steps. Information about a problem given in the context is often ignored. Though this is classified as a question on probability, it would appear to be a question relating to linking probability to a frequentist approach and then performing some arithmetical calculations.

4 At level 8, a question entitled *Pots* assessed the probability of combined events.

> In pottery, when you make a clay object like a vase, you have to harden it by firing. This process leads to a permanent form of the object but it might crack in the firing process. The item on Pots dealt with such a situation, see QCA (pp. 17).
>
> The probability of a pot cracking in the firing process is 0.03. Two pots are fired.
>
> (a) What is the probability that both pots crack.
>
> (b) What is the probability that only one of the two pots cracks.

Pupils were given the probability 0.03 of a pot cracking when fired. In part (a) they were asked to calculate the probability of both pots cracking when two were fired. The common incorrect response in part (a) was that of 0.06, obtained by doubling 0.03 rather than squaring it. This error was made by almost 40% of pupils at level 7.

This indicates quite a deep lack of conceptual understanding with children opting for a numerical calculation based on techniques which have been presented but not explained in detail or linked to core ideas. However, of those pupils at level 8 (reached by under 10% of the cohort), over 65% gained the mark; and a further 20% had shown the calculation 0.03×0.03 in their working although their final answers were incorrect.

Part (b) asked for the probability of only one pot cracking when two were fired. This was found difficult and fewer than 10% of pupils gaining level 8 overall were awarded both marks. However, a further 40% gained one of the two marks for an answer of0.0291 or 0.029, which indicates that pupils had forgotten to take order into account. The common incorrect response from pupils at level 7 was 0.03, based on the calculation 0.03×1. It is not clear why multiplication is used; it would appear to be use of a technique without justification.